Answers and Hints to Game Theory Evolving:

A Problem-Centered Introduction to Modeling
Strategic Behavior

Herbert Gintis

Princeton University Press
Princeton, New Jersey

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Leading from Strength: Eliminating Dominated Strategies

2.1 Introduction

2.2 Dominated Strategies

2.3 Backward Induction: Pruning the Game Tree

2.4 Eliminating Dominated Strategies

a. $N_2 < J_2, C_1 < N_1, J_2 < C_2, N_1 < J_1$
b. $C > D, e > a, B > E, c > b, B > A, c > d, B < C$.

2.5 Concepts and Definitions

2.6 The Prisoner's Dilemma

2.7 An Armaments Game

2.8 Second-Price Auction

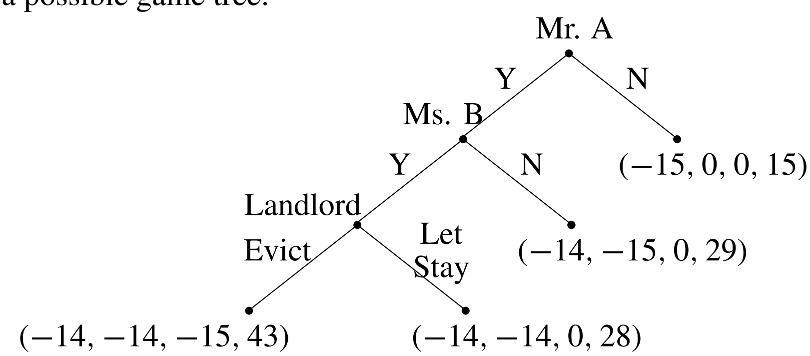
Suppose first you win, and let v_s be the second-highest bid. If you had bid more than v_i, you still would have won, and your gain would still be the same, namely $v_i - v_s \geq 0$. If you had bid lower than v_i, there are three subcases: you could have bid more than, equal to, or less than v_s. If you had bid more than v_s, you would have had the same payoff, $v_i - v_s$. If you had bid equal to v_s, you could have lost the auction in the playoff among the equally high bidders, and if you had bid less than v_s, you certainly would have lost the auction. Hence, nothing beats bidding v_i in case you win.

But suppose you bid v_i and lost. Let v_h be the highest bid and v_s be the second-highest bid. Since you lost, your payoff is zero, so if you had bid less than v_i, you would still have lost, so you couldn't improve your payoff this way. If had you bid more than v_i, it wouldn't matter unless you had bid enough to win the auction, in which case your loss would have been $v_s - v_i$. Since $v_i \neq v_h$, we must have $v_i \leq v_s$, as v_s is the second-highest offer. Thus, you could not have made a positive gain by bidding higher than v_i. Hence, bidding v_i is a best response to any set of bids by the other players.

2.9 Unraveling the Mystery of Kidnapping

2.10 The Landlord and the Eviction Notice

Here is a possible game tree:



2.11 Hagar's Battles

Each side should deploy its troops to the most valuable battlefields. To see this suppose player 1 does not. Let x_j be the highest value battlefield unoccupied by player 1, and let x_i be the lowest value battlefield occupied by player 1. What does player 1 gain by switching a soldier from x_i to x_j? If both are occupied by player 2, there is no change. If neither is occupied by player 2, player 1 gains $a_j - a_i > 0$. If player 2 occupies x_j but not x_i, player 1 loses a_i by switching, and player 2 loses a_j, so player 1 gains $a_j - a_i > 0$. Similarly if player 2 occupies x_i but not x_j.

Another explanation: Suppose you occupy a_i but not a_j, where $a_j > a_i$. The figure below shows that the gain from switching from a_i to a_j is positive in all contingencies.

Enemy Occupies

	a_i not a_j	a_j not a_i	a_i and a_j	neither
loss	lose i	lose i	lose i	lose i
gain	gain j	gain j	gain j	gain j
net gain	$j - i$	$j - i$	$j - i$	$j - i$

2.12 Football Strategy

2.13 A Military Strategy Game

First we can eliminate all Country I strategies that do not arrive at A. This leaves six strategies, which we can label fcb, feb, fed, hed, heb, and hgd. We can also eliminate all Country A strategies that stay at A at any time, or that hit h or f. This leaves the six strategies bcb,beb,bed,ded,deb,dgd. Here is the payoff matrix:

	bcb	beb	bed	ded	deb	dgd
fcb	−1	−1	1	1	−1	1
feb	−1	−1	−1	−1	−1	1
fed	1	−1	−1	−1	−1	−1
hed	1	−1	−1	−1	−1	−1
heb	−1	−1	−1	−1	−1	1
hgd	1	1	−1	−1	1	−1

Now feb is weakly dominated by fcb, as is heb. Moreover, we see that fed and hed are weakly dominated by hgd. Thus there are two remaining strategies for Country I, “south” (hgd) and “north” (fcb).

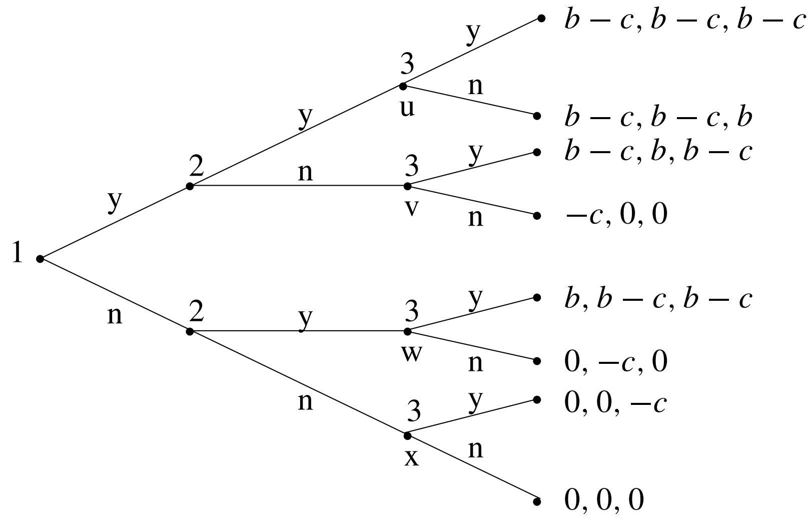
Also bcb is dominated by beb and dgd is dominated by ded, so we may drop them. Moreover, beb and deb are the same “patrol north”, while bed and ded are the same “patrol south.” This gives us the following reduced game:

	patrol north	patrol south
attack north	−1,1	1,−1
attack south	1,−1	−1,1

So this complicated game is just the heads-tails game, which we will finish solving when we do mixed strategy equilibria!

2.14 Strategic Voting

We can solve this by pruning the game tree. We find that player 1 chooses "no,' and players 2 and 3 choose "yes," with payoff $(b, b-c, b-c)$. It is best to go first. The game tree is:



Note that this does not give a full specification of the strategies, since player 2 has four strategies and player 3 has sixteen strategies. The above description says only what players 2 and 3 do "along the game path," i.e., as the game is actually played.

To describe the Nash equilibrium in full, let us write player 3's strategies as "uvwx," where u, v, w, and x are each either y ("yes") or n ("no") and indicate the choice at the corresponding node in the game tree, starting from the top. Then the third player's choice is nyyn. Similarly, player 2's choice is ny, and the first player's is, of course, n.

If player 3 chooses nnnn, player 2 chooses yn, and player 1 chooses y, we have another Nash equilibrium (check it out!), in which player 3 now gets b and the other two get $b-c$. The equilibrium is strange because it means that player 3 should make suboptimal choices at nodes v and w—he says he will choose "no" but in fact he will choose "yes" at these nodes, since this gives him a higher payoff. The strategy nnnn is called an *incredible threat*, because it involves player 3 threatening to do something that he in fact will not do when it comes time to do it. But if the others believe him, he will never have to carry out his threat! We say such a Nash equilibrium violates *subgame perfection*, a phenomenon of great importance that we have touched upon in the Big Monkey and Little Monkey game (§1.2), and we will take it up systematically in chapter 7.

2.15 Eliminating Dominated Strategies *ad Absurdum*

2.16 Poker with Bluffing

2.17 Cooperation in the Repeated Prisoner's Dilemma

2.18 Dominance Solvable Games

3

Behavioral Decision Theory

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3.6 Compound Lotteries

3.6.1 Where's Jack?

Let A be the event "Manny wins on the first round," B the event "Moe wins on the second round," and C the event "Manny wins on the third round." We have $p(A) = 0.3$, $p(B) = (1 - p(A))(0.5) = 0.35$, and $p(C) = (1 - p(B) - p(A))(0.4) = 0.14$. Thus, the probability Manny wins is $p(A) + p(C) = 0.44$, so Manny's expected payoff is \$44.

3.7 The St. Petersburg Paradox

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Behavioral Game Theory

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Playing It Straight: Pure Strategy Nash Equilibria

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5.3 Pure Coordination Games: Experimental Evidence

5.4 Pick Any Number

5.5 Competition on Main Street

5.6 Twin Sisters

5.7 Variations on Duopoly

5.8 The Tobacco Market

a. Let's not use numbers until we need to. We can write $p = a - bq$, where $q = q_1 + q_2 + q_3$, and q_i is the amount sent to market by farmer i. Farmer 1 maximizes $pq_1 = (a - bq)q_1$. If there is an interior solution, the first-order condition on q_1 must satisfy

$$a - b(q_2 + q_3) - 2bq_1 = 0.$$

If all farmers ship the same amount of tobacco, then $q_2 = q_3 = q_1$, so this equation becomes $4bq_1 = a$, which gives $q_1 = q_2 = q_3 = a/4b$, and $q = 3a/4b$, so $p = a/4$. The profit of each farmer is $pq = a^2/16b$. In our case $a = 10$ and $b = 1/100000$, so the price is 2.50 per pound, and each farmer ships 250,000 pounds and discards the rest. The price support doesn't matter, since $p > 0.25$. Each farmer has profit 625,000.

If the second and third farmers send their whole crop to market, then $q_2 + q_3 = 1,200,000$. In this case even if farmer 1 shipped nothing, the market price would be $10 - 1,200,000/100,000 = -2 < 0.25$, so the price support would kick in. Farmer 1 should then also ship all his tobacco at 0.25 per pound, and each farmer has profit 150,000.

b. You can check that there are no other Nash equilibria. If one farmer sends all his crop to market, the other two would each send 400,000/3 pounds to market. But then the first farmer would gain by sending less to market.

5.9 Price-Matching as Tacit Collusion

5.10 The Klingons and the Snarks

Suppose the Klingons choose a common rate r of consumption. Then each eats 500 snarks, and each has payoff

$$u = 2000 + 50r - r^2.$$

Setting the derivative u' to zero, we get $r = 25$, so each has utility $u = 2000 + 50(25) - 25^2 = 2625$.

Now suppose they choose their rates separately. Then

$$u_1 = \frac{4000r_1}{r_1 + r_2} + 50r_1 - r_1^2.$$

Setting the derivative of this to zero, we get the first-order condition

$$\frac{\partial u_1}{\partial r_1} = \frac{4000r_2}{(r_1 + r_2)^2} + 50 - 2r_1 = 0,$$

and a symmetrical condition holds for the second Klingon:

$$\frac{\partial u_2}{\partial r_2} = \frac{4000r_1}{(r_1 + r_2)^2} + 50 - 2r_2 = 0.$$

These two imply

$$\frac{r_2}{r_1} = \frac{r_1 - 25}{r_2 - 25},$$

which has solutions $r_1 = r_2$ and $r_1 + r_2 = 25$. The latter, however, cannot satisfy the first-order conditions. Setting $r_1 = r_2$, we get

$$\frac{4000}{4r_1} + 50 - 2r_1 = 0,$$

or $1000/r_1 + 50 - 2r_1 = 0$. This is a quadratic that is easy to solve. Multiply by r_1, getting $2r_1^2 - 50r_1 - 1000 = 0$, with solution $r = (50 + \sqrt{(2500 + 8000)})/4 = 38.12$. So the Klingons eat about 50% faster than they would if they cooperated! Their utility is now $u = 2000 + 50r_1 - r_1^2 = 2452.87$, lower than if they cooperated.

5.11 Chess—The Trivial Pastime

a. The game in §5.10 is not finite and is not a game of perfect information. The game of Matching Pennies (§6.8) has no pure strategy Nash equilibrium.
b. We will have to prove something more general. Let's call a game *Chessian* if it is a finite game of perfect information in which players take turns, and the outcome is either (win,lose), (lose,win), or (draw,draw), where win is preferred to draw, and draw is preferred to lose. Let us call a game *certain* if it has a solution in pure strategies. If a Chessian game is certain, then clearly either one player has a winning strategy, or both players can force a draw. Suppose there were a Chessian game that is not certain. Then there must be a *smallest* Chessian game that is not certain (i.e., one with fewest nodes). Suppose this has k nodes. Clearly, $k > 1$, since it is obvious that a Chessian game with one node is certain. Take any node all of whose child nodes are terminal nodes (why must this exist?). Call this node A. Suppose Red (player 1) chooses at A (the argument is similar if Black chooses). If one of the terminal nodes from A is (win,lose), label A (lose,win); if all of the terminal nodes from A are (lose,win), label A (win,lose); otherwise label A (draw,draw). Now erase the branches from A, along with their terminal nodes. Now we have a new, smaller, Chessian game, which is certain, by our induction assumption. It is easy to see that if Red has a winning strategy in the smaller game, it can be extended to a winning strategy in the larger game. Similarly, if Black has a winning strategy in the smaller game, it can be extended to a winning strategy in the larger game. Finally, if both players can force a draw in the smaller game, their respective strategies must force a draw in the larger game.

5.12 The Samaritan's Dilemma

The father's first-order condition is

$$U_t(t,s) = -u'(y-t) + \alpha\delta v_2'(s(1+r)+t) = 0,$$

and the father's second-order condition is

$$U_{tt} = u''(y-t) + \alpha\delta v_2''(s(1+r)+t) < 0.$$

If we treat t as a function of s (backward induction!), then the equation $U_t(t(s),s) = 0$ is an identity, so we can differentiate totally with respect to s, getting

$$U_{tt}\frac{dt}{ds} + U_{ts} = 0.$$

But $U_{ts} = \alpha\delta(1+r)v_2'' < 0$, so $t'(s) < 0$; i.e., the more the daughter saves, the less she gets from her father in the second period.

The daughter's first-order condition is

$$v_s(s,t) = -v_1' + \delta v_2'(1+r+t'(s)) = 0.$$

Suppose the daughter's optimal s is s^*, and so the father's transfer is $t^* = t(s^*)$. If the father precommits to t^*, then $t'(s) = 0$ would hold in the daughter's first-order condition. Therefore, in this case $v_s(s^*, t^*) > 0$, so the daughter is better off by increasing s to some $s^{**} > s^*$, and hence the father is better off as well, since he is a partial altruist.

For the example, if $u(\cdot) = v_1(\cdot) = v_2(\cdot) = \ln(\cdot)$, then it is straightforward to check that

$$t^* = \frac{y(1+\alpha\delta(1+\delta)) - \delta(1+r)z}{(1+\delta)(1+\alpha\delta)}$$
$$s^* = \frac{\delta(1+r)z - y}{(1+r)(1+\delta)}.$$

If the father can precommit, solving the two first-order conditions for maximizing $U(t,s)$ gives

$$t^f = \frac{\alpha(1+\delta)y - (1+r)z}{1+\alpha+\alpha\delta},$$
$$s^f = \frac{(1+r)(1+\alpha\delta)z - \alpha y}{(1+r)(1+\alpha+\alpha\delta)}.$$

We then find

$$t^* - t^f = \frac{y + (1+r)z}{(1+\delta)(1+\alpha\delta)(1+\alpha+\alpha\delta)} > 0,$$
$$s^f - s^* = \frac{y + (1+r)z}{(1+r)(1+\delta)(1+\alpha+\alpha\delta)} > 0.$$

5.13 The Rotten Kid Theorem

a. Mom's objective function is

$$V(t,a) = u(y(a) - t) + \alpha v(z(a) + t),$$

so her first-order condition is

$$V_t(t,a) = -u'(y(a)-t) + \alpha v'(z(a)+t) = 0$$

If we treat t as a function of a in the above equation (this is where backward induction comes in!), it becomes an identity, so we can differentiate with respect to a, getting

$$-u''(y'-t') + \alpha v''(z'+t') = 0. \tag{A5.1}$$

Therefore, $z'+t'=0$ implies $y'-t'=y'+z'=0$. Thus the first-order conditions for the maximization of $z+t$ and $z+y$ have the same solutions.

b. Note that since a satisfies $z'(a)+y'(a)=0$, a does not change when α changes. Differentiating the first-order condition $V_t(t(\alpha))=0$ totally with respect to α, we get

$$V_{tt}\frac{dt}{d\alpha} + V_{t\alpha} = 0.$$

Now $V_{tt} < 0$ by the second-order condition for a maximum, and

$$V_{t\alpha} = v' > 0,$$

which proves that $dt/d\alpha > 0$. Since a does not depend on $\hat{y}$, differentiating $V_t(t(y)) = 0$ totally with respect to $\hat{y}$, we get

$$V_{tt}\frac{dt}{d\hat{y}} + V_{t\hat{y}} = 0.$$

But $V_{t\hat{y}} = -u'' > 0$ so $dt/d\hat{y} > 0$.

c. Suppose t remains positive as $\alpha \to 0$. Then v' remains bounded, so $\alpha v' \to 0$. From the first-order condition, this means $u' \to 0$, so $y - t \to \infty$. But y is constant, since a maximizes $y+z$, which does not depend on α. Thus $t \to -\infty$.

5.14 Payoffs in Games Where Nature Moves

5.15 The Illogic of Conflict Escalation

We do only the last part of the problem. Each player in effect faces a decision problem against Nature where staying in round k wins v with probability p, loses k with probability $p(1-p)$, and leads to round $k+2$ with probability $(1-p)^2$.

Dropping in round k for $k > 2$ costs $k-2$. The pure strategies are s_k, which says "bid until round k, then drop if you have not already won or lost." We calculate the expected payoff to s_k. Consider player 1, who bids first. The probability that the game terminates spontaneously at round n is $(1-p)^{n-1}p$. Player 1 wins $v-n$ if the game terminates spontaneously on an odd round $n < k$, loses $n-1$ if the game terminates spontaneously on an even round $n < k$, and loses $k-2$ if the game continues to round k. The expected return from winning is 0 for $k = 1$ and for $k \geq 3$ is given by

$$v_{+k} = (v-1)p + (v-3)p(1-p)^2 + \ldots + (v-k+2)p(1-p)^{k-3}$$

Note that this also holds for $k = 1$. The expected cost of losing by spontaneous termination is

$$v_{-k} = p(1-p) + 3p(1-p)^3 + \ldots + (k-2)p(1-p)^{k-2}.$$

Finally, the expected loss from dropping out at round k is $k-2$ times the probability of reaching round k, or $(k-2)(1-p)^{k-1}$. Thus the total value of strategy s_k for $k > 1$ is $v_{+k} - v_{-k} - (k-2)(1-p)^{k-1}$. This can be summed by hand, but it is very tedious. Mathematica calculated it for me, giving the value v_k of stopping on round k equal (after some simplification) to

$$v_k = \frac{p^3 - p^2(v+3) + 4p + (vp-2)(1-(1-p)^k)}{(2-p)(1-p)p}.$$

The derivative of v_k is $-\ln(1-p)(pv-2)(1-p)^{k-1}/p(2-p)$, which is positive if $v > 2/p$. Since $v_1 = 1$, this shows that Player 1 should never stop. The analysis for player 2 is similar.

5.16 Agent-Based Modeling

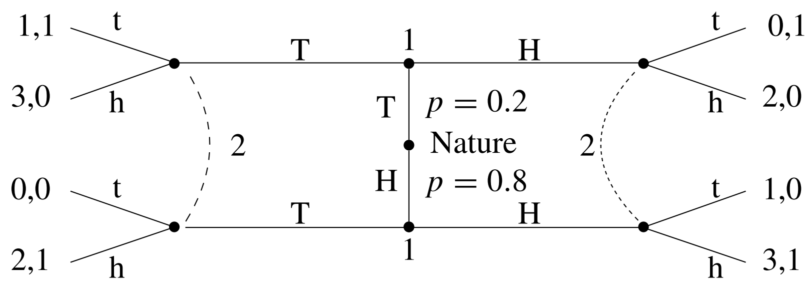
5.17 Nature in Action: No-Draw, High-Low Poker

5.17.1 Simulating No-Draw High-Low Poker

5.18 Markets as Disciplining Devices: Allied Widgets

5.19 The Truth Game

b. Here is the game tree, written "sideways":



Player 1 has strategy set {HH,HT,TH,TT}, where HH means announce H if you see H, announce H if you see T, HT means announce H if you see H, announce T if you see T, etc. Thus, HH = always say H, HT = tell the truth, TH = lie, TT = always say T. Player 2 has strategy set {hh,ht,th,tt}, where hh means say h if you are told h, and say h if you are told t, ht means say h if you are told h and say t if you are told t, etc. Thus, hh means always say h, ht means trust player 1, th means distrust player 1, and tt means always say t.

c. Here are the payoffs to the two cases, according to Nature's choice:

	hh	ht	th	tt
HH	2,0	2,0	0,1	0,1
HT	3,0	1,1	3,0	1,1
TH	2,0	2,0	0,1	0,1
TT	3,0	1,1	3,0	1,1

Payoff when coin is T

	hh	ht	th	tt
HH	3,1	3,1	1,0	1,0
HT	3,1	3,1	1,0	1,0
TH	2,1	0,0	2,1	0,0
TT	2,1	0,0	2,1	0,0

Payoff when coin is H

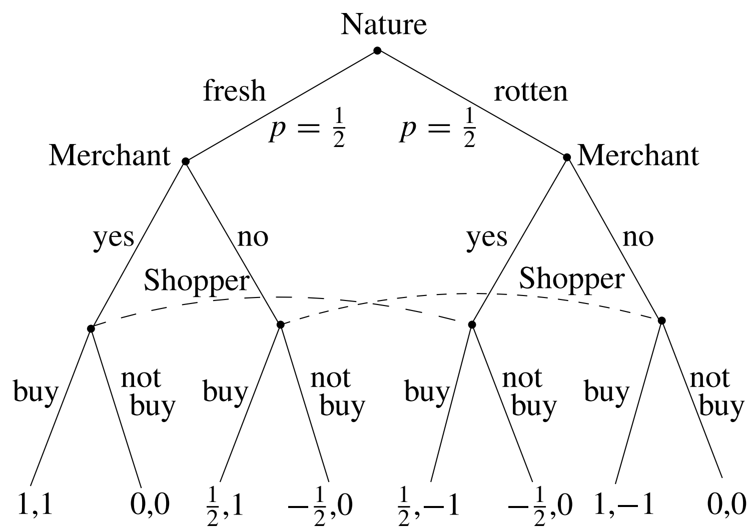
The actual payoff matrix is $0.2 \times$ first matrix $+ 0.8 \times$ second:

	hh	ht	th	tt
HH	2.8,0.8	2.8,0.8	0.8,0.2	0.8,0.2
HT	3.0,0.8	2.6,1.0	1.4,0.0	1.0,0.2
TH	2.0,0.8	0.4,0.0	1.6,1.0	0.0,0.2
TT	2.2,0.8	0.2,0.8	2.2,0.8	0.2,0.2

d. tt is dominated by hh, so we can drop tt. There are no other dominated pure strategies.

e. (TT,th) and (HH,ht) are both Nash. In the first, 1 always says T and 2 assumes 1 lies; in the second, 1 always says H and 2 always believes 1. The first equilibrium is Pareto-inferior to the second.

5.20 The Shopper and the Fish Merchant



Here is the normal form for each of the two cases Good Fish/Bad Fish, and their expected value:

	bb	bn	nb	nn
yy	1,1	1,1	0,0	0,0
yn	1,1	1,1	0,0	0,0
ny	$\frac{1}{2}$,1	$-\frac{1}{2}$,0	$\frac{1}{2}$,1	$-\frac{1}{2}$,0
nn	$\frac{1}{2}$,1	$-\frac{1}{2}$,0	$\frac{1}{2}$,1	$-\frac{1}{2}$,0

Good Fish

	bb	bn	nb	nn
yy	$\frac{1}{2}, -1$	$\frac{1}{2}, -1$	$-\frac{1}{2}, 0$	$-\frac{1}{2}, 0$
yn	$1, -1$	0, 0	$1, -1$	0, 0
ny	$\frac{1}{2}, -1$	$\frac{1}{2}, -1$	$-\frac{1}{2}, 0$	$-\frac{1}{2}, 0$
nn	$1, -1$	0, 0	$1, -1$	0, 0

Bad Fish

	bb	bn	nb	nn
yy	$\frac{3}{4}, 0$	$\frac{3}{4}$,0	$-\frac{1}{4}$,0	$-\frac{1}{4}, 0$
yn	1,0	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}$,$-\frac{1}{2}$	0,0
ny	$\frac{1}{2}, 0$	$0, -\frac{1}{2}$	$0, \frac{1}{2}$	$-\frac{1}{2}, 0$
nn	$\frac{3}{4}$,0	$-\frac{1}{4}$,0	$\frac{3}{4}$,0	$-\frac{1}{4}$,0

$\frac{1}{2}$ Good Fish + $\frac{1}{2}$ Bad Fish

Applying the successive elimination of dominated strategies, we have yn dominates ny, after which bn dominates bb, nb, and nn. But then yy dominates yn and nn. Thus, a Nash equilibrium is yy/bn: the merchant says the fish is good no matter what, and the buyer believes him. Since some of the eliminated strategies were only weakly dominated, there could be other Nash equilibria, and we should check for this. We find that another is for the seller to use pure strategy nn and the buyer to use pure strategy nb. Note that this equilibrium only works if the buyer is a "nice guy" in the sense of choosing among equally good responses that maximizes the payoff to the seller. The equilibrium yy/bn does not have this drawback.

5.21 Fathers and Sons

5.22 The Women of Sevitan

There are six philandering husbands, and all are branded on the sixth day. The reason this happens is that the women all knew there was at least one philanderer, but they didn't know that the other women knew this. That is, the fact that each woman knows there is a philanders is not known to the other five women until six rounds of the game are played. Here is a sketch of the general argument.

Suppose there is one philanderer. Then all women but one would see one philanderer. The wife of the philanderer would see no philanderers. Thus when the visitor informs the village that there is a philanderer, the wife of the philanderer knows it must be her husband, so she would brand him on night 1.

Suppose there are two philanderers: All women but two see two philanderers, but the wives of the philanderers each sees one philanderer. Each of deceived wives reasons that if her husband were not a philanderer, she would be in the previous case. But since no wife brands on night one, both deceived wives must brand on night 2. The new information in this case is that each of the deceived wives knew there was one philanderer, but did not know that the other deceived wife knew there was at least one, until the second night.

Suppose there are three philanderers. All women but three see three philanderers. Each of the three deceived wives sees two philanderers. Each of them reasons that if her husband were not a philanderer, this would be the previous case. But that is impossible after night 2. Thus, each deceived wife brands on night 3. The new information in this case is that each of the deceived wives knew there were at least two philanderers, and hence knew that each of the others knew there was at least one philanderer. Each deceived woman thinks, "If I am not deceived, then I have learned that each of the two deceived women knows the other knows there is at least one philanderer." This is new information. After the first night, this new information implies each deceived woman knows she is deceived.

This argument can be extended any number of rounds. Since no husband is branded in the first five nights, they must all be philanderers, and hence will be branded on the sixth night.

6

Catching 'em Off Guard: Mixed Strategy Nash Equilibria

6.1 Introduction

6.2 Mixed Strategies: Basic Definitions

6.3 The Algebra of Mixed Strategies

6.4 The Fundamental Theorem

6.5 Solving for Mixed Strategy Nash Equilibria

6.6 Reviewing the Terminology

6.7 Battle of the Sexes

b. Let α be the probability of Alfredo going to the opera, and let β be the probability of Elisabetta going to the opera. Since in a mixed strategy equilibrium, the payoff to gambling and opera must be equal for Alfredo, we must have

$$\beta = 2(1-\beta),$$

which implies $\beta = 2/3$. Since the payoff to gambling and opera must also be equal for Elisabetta, we must have

$$2\alpha = 1-\alpha,$$

so $\alpha = 1/3$. The payoff of the game to each is then

$$\frac{2}{9}(1,2) + \frac{5}{9}(0,0) + \frac{2}{9}(2,1) = \left(\frac{2}{3},\frac{2}{3}\right),$$

since both go gambling $(1/3)(2/3) = 2/9$ of the time, both go to the opera $(1/3)(2/3) = 2/9$ of the time, and otherwise they miss each other.

c. Both do better if they can coordinate, since (2,1) and (1,2) are both better than (2/3,2/3).

Note that we get the same answer if we find the Nash equilibrium by finding the intersection of the players' best response function. To see this, note that the payoffs to the two players are

$$\pi_1 = \alpha\beta + 2(1-\alpha)(1-\beta) = 3\alpha\beta - 2\alpha - 2\beta + 2$$
$$\pi_2 = 2\alpha\beta + (1-\alpha)(1-\beta) = 3\alpha\beta - \alpha - \beta + 1.$$

Thus,

$$\frac{\partial \pi_1}{\partial \alpha} = 3\beta - 2 \begin{cases} > 0 & \text{if } \beta > 2/3 \\ = 0 & \text{if } \beta = 2/3 \\ < 0 & \text{if } \beta < 2/3 \end{cases},$$

so the optimal α is given by

$$\alpha = \begin{cases} 1 & \text{if } \beta > 2/3 \\ [0,1] & \text{if } \beta = 2/3 \\ 0 & \text{if } \beta < 2/3 \end{cases}.$$

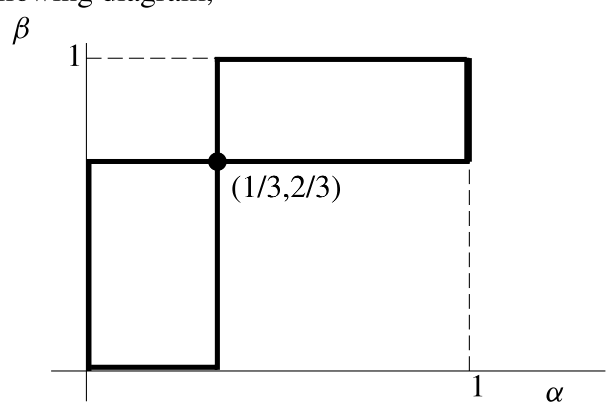
Similarly,

$$\frac{\partial \pi_2}{\partial \alpha} = 3\alpha - 1 \begin{cases} > 0 & \text{if } \alpha > 1/3 \\ = 0 & \text{if } \alpha = 1/3 \\ < 0 & \text{if } \alpha < 1/3 \end{cases},$$

so the optimal β is given by

$$\beta = \begin{cases} 1 & \text{if } \alpha > 1/3 \\ [0,1] & \text{if } \alpha = 1/3 \\ 0 & \text{if } \alpha < 1/3 \end{cases}.$$

This gives the following diagram,



6.8 Matching Pennies

Clearly there are no pure strategy equilibria. So suppose player 2 uses the mixed strategy σ that consists of playing c_1 (one finger) with probability α and c_2 (two fingers) with probability $1-\alpha$. We write this as $\sigma = \alpha c_1 + (1-\alpha)c_2$. If player 1 uses both r_1 (one finger) and r_2 (two fingers) with positive probability, they both must have the same payoff against σ, or else player 1 should drop the lower-payoff strategy and use only the higher-payoff strategy. The payoff to r_1 against σ is $\alpha \cdot 1 + (1-\alpha) \cdot -1 = 2\alpha - 1$, and the payoff to r_2 against σ is $\alpha \cdot -1 + (1-\alpha) \cdot 1 = 1 - 2\alpha$. If these are equal, then $\alpha = 1/2$. A similar reasoning shows that player 1 chooses each strategy with probability 1/2. The expected payoff to player 1 (often called the "value of the game" to player 1) is then $2\alpha - 1 = 1 - 2\alpha = 0$, and the same is true for player 2.

6.9 The Hawk-Dove Game

6.10 Lions and Antelope

Let BA be the strategy "hunt Big Antelope," and let LA be the strategy "hunt Little Antelope." Here is the normal form game:

	BA	LA
BA	$c_b/2$,$c_b/2$	c_b,c_l
LA	c_l,c_b	$c_l/2$,$c_l/2$

If (BA,BA) is to be a pure strategy equilibrium, it is necessary that $c_b/2 \geq c_l$, and it is easy to see that this condition is also sufficient. Since $c_b > c_l$, it is easy to see that (LA,LA) is not Nash.

To find the mixed strategy equilibrium assuming (BA,BA) is not Nash, suppose $c_b < 2c_l$. Let α be the probability a lion uses BA. Then the payoff to the other lion from using BA is

$$\alpha \frac{c_b}{2} + (1-\alpha)c_b = c_b - \alpha \frac{c_b}{2},$$

and the payoff to using LA is

$$\alpha c_l + (1-\alpha)\frac{c_l}{2} = (1+\alpha)\frac{c_l}{2}.$$

Equating these two, we get

$$\alpha = \frac{2c_b - c_l}{c_b + c_l}.$$

The payoff to the lions is then equal to the payoff to BA, which is

$$\begin{aligned} c_b - \alpha\frac{c_b}{2} &= c_b\left(1 - \frac{\alpha}{2}\right) \\ &= c_b\frac{3c_l}{2(c_b + c_l)}. \end{aligned}$$

It is easy to check that the fraction above is greater than 1/2, so they should play the mixed strategy.

One can also calculate the expected payoff using the payoff to LA instead of the payoff to BA:

$$\begin{aligned} (1+\alpha)\frac{c_l}{2} &= \left(\frac{3c_b}{c_b + c_l}\right)\frac{c_l}{2} \\ &= c_l\frac{3c_b}{2(c_b + c_l)}, \end{aligned}$$

which is the same.

6.11 A Patent Race

6.12 Tennis Strategy

	b_r	f_r
b_s	.4, .6	.7, .3
f_s	.8, .2	.1, .9

$$\begin{aligned} \sigma &= \alpha b_r + (1-\alpha) f_r \\ \tau &= \beta b_s + (1-\beta) f_s \\ \pi_{b_s} &= \alpha\pi_1(b_s, b_r) + (1-\alpha)\pi_1(b_s, f_r) \\ &\quad \text{where } \pi_1(b_s, b_r) = \text{Server's payoff to } b_s, b_r \\ &= .4\alpha + .7(1-\alpha) = .7 - .3\alpha \\ \pi_{f_s} &= \alpha\pi_1(f_s, b_r) + (1-\alpha)\pi_1(f_s, f_r) \\ &= .8\alpha + .1(1-\alpha) = .1 + .7\alpha \end{aligned}$$

$$.7 - .3\alpha = .1 + .7\alpha \Rightarrow \boxed{\alpha = 3/5}$$
$$\pi_{b_r} = \beta\pi_2(b_s, b_r) + (1-\beta)\pi_2(f_s, b_r)$$
$$= .6\beta + .2(1-\beta) = .2 + .4\beta$$
$$\pi_{f_r} = \beta\pi_2(b_s, f_r) + (1-\beta)\pi_2(f_s, f_r)$$
$$= .3\beta + .9(1-\beta) = .9 - .6\beta$$
$$.2 + .4\beta = .9 - .6\beta \Rightarrow \boxed{\beta = 7/10}$$

Payoffs to Players:

$$\pi_1 = .4 \cdot \frac{3}{5} + .7 \cdot \frac{2}{5} = .52$$
$$\pi_2 = .6 \cdot \frac{7}{10} + .2 \cdot \frac{3}{10} = .48.$$

6.13 Preservation of Ecology Game

The first two parts are easy. For the third and fourth, suppose x, y and z are the probabilities the three firms purify. If firm 3 purifies, its expected payoff is $-xy - x(1-y) - y(1-x) - 4(1-x)(1-y)$. If firm 3 pollutes, its payoff is $-3x(1-y) - 3(1-x)y - 3(1-x)(1-y)$. If firm 3 is to use a mixed strategy, these must be equal, so after simplification we have $(1-3x)(1-3y) = 3xy$. Solving, and repeating for the other two firms, we get the two desired solutions.

6.14 Hard Love

6.15 Advertising Game

a. It is clear that the strategies described are Nash. We will show that these are the *only* Nash equilibria in which at least one firm uses a pure strategy. Suppose first that firm 1 chooses the pure strategy M (morning). If both firms 2 and 3 choose mixed strategies, then one of them could gain by shifting to pure strategy E (evening). To see this, let the two mixed strategies be $\alpha M + (1-\alpha)E$ for firm 2 and $\beta M + (1-\beta)E$ for firm 3. Let $\pi_i(s_1s_2s_3)$ be the payoff to player i when the three firms use pure strategies $s_1s_2s_3$. Then, the payoff to M for firm 2 is

$$\pi_2 = \alpha\beta\pi_2(MMM) + \alpha(1-\beta)\pi_2(MME) + (1-\alpha)\beta\pi_2(MEM)$$

$$
\begin{aligned}
&+ (1-\alpha)(1-\beta)\pi_2(MEE)\\
&= \alpha\beta(0) + \alpha(1-\beta)(0) + (1-\alpha)\beta(2) + (1-\alpha)(1-\beta)(0)\\
&= 2(1-\alpha)\beta.
\end{aligned}
$$

Since $0 < \beta$ by definition, this is maximized by choosing $\alpha = 0$, so firm 2 should use pure strategy E. This contradicts our assumption that both firms 1 and 2 use mixed strategies.

A similar argument holds if firm 1 uses pure strategy E. We conclude that if firm 1 uses a pure strategy, at least one of the other two firms will use a pure strategy. The firm that does will not use the same pure strategy as firm 1, since this would not be a best response. Therefore, two firms use opposite pure strategies, and it doesn't matter what the third firm does.

Now we repeat the whole analysis assuming firm 2 uses a pure strategy, with clearly the same outcome. Then, we do it again for firm 3.

This proves that if one firm uses a pure strategy, at least two firms use a pure strategy, which concludes this part of the problem.

b. Let x, y, and z be the probabilities of advertising in the morning for firms 1, 2, and 3. The expected return to 1 of advertising in the morning is $(1-y)(1-z)$, and in the evening it is $2yz$. If these are equal, any choice of x for firm 1 is Nash. But equality means $1 - y - z - yz = 0$, or $y = (1-z)/(1+z)$. Now repeat for firms 2 and 3, giving the equalities $y = (1-z)/(1+z)$ and $z = (1-x)/(1+x)$. Solving simultaneously, we get $x = y = z = \sqrt{2} - 1$. To see this, substitute $y = (1-z)/(1+z)$ in $x = (1-y)/(1+y)$, getting

$$
x = \frac{1-y}{1+y} = \frac{1 - \frac{1-z}{1+z}}{1 + \frac{1-z}{1+z}} = z.
$$

Thus, $x = (1-x)/(1+x)$, which is a simple quadratic equation, the only root of which between 0 and 1 is $\sqrt{2} - 1$. Thus, this is Nash.

To show that there are no other Nash equilibria, suppose $0 < x < 1$ and $0 < y < 1$. We must show $0 < z < 1$, which reproduces equilibrium (b). But $0 < x < 1$ implies $(1+y)(1+z) = 2$ (*why?*), and $0 < y < 1$ implies $(1+x)(1+z) = 2$. If $z = 0$, then $x = y = 1$, which we assumed is not the case. If $z = 1$ then $x = y = 0$, which is also not the case. This proves it.

6.16 Robin Hood and Little John

a. The payoff matrix is as follows:

	G	W
G	$-\delta - \tau_{lj}/2$ $-\delta - \tau_r/2$	0 $-\tau_r$
W	$-\tau_{lj}$ 0	$-\epsilon - \tau_{lj}/2$ $-\epsilon - \tau_r/2$

b. The pure Nash equilibria are:

$$\begin{aligned} GG&: \quad \tau_r, \tau_{lj} \geq 2\delta \\ WG&: \quad 2\delta \geq \tau_{lj} \\ GW&: \quad 2\delta \geq \tau_r. \end{aligned}$$

There is also a mixed strategy with α_r and α_{lj} being the probabilities of Robin Hood and Little John going

$$\alpha_{lj} = \frac{\epsilon + \tau_{lj}/2}{\epsilon + \delta}, \qquad \alpha_r = \frac{\epsilon + \tau_r/2}{\epsilon + \delta}$$

for $2\delta > \tau_r, \tau_{lj}$.

d. Suppose $\tau_r > \tau_{lj}$. Then, the socially optimal δ is any δ satisfying $\tau_r > 2\delta > \tau_{lj}$, since in this case it never pays to fight. The cost of crossing the bridge is τ_{lj} (or $\tau_r + 2\tau_{lj}$), including the crossing-time itself. Of course, this makes Robin Hood wait all the time. He might prefer to lower or raise the costs of fighting. Will he? The payoff to the game to the players when $\tau_r > 2\delta > \tau_{lj}$ is $(-\tau_{lj}, 0)$.

Suppose Robin Hood can shift to lower-cost confrontation: we lower δ so $\tau_r > \tau_{lj} > 2\delta$. Then, GG is dominant, and the gain to the two players is $(-\delta - \tau_{lj}/2, -\delta - \tau_r/2)$, which is better for Robin Hood if and only if $-\tau_{lj} < -\delta - \tau_{lj}/2$, or $2\delta < \tau_{lj}$, *which is true*! Therefore, *Robin Hood gains if he can shift to a lower-cost form of fighting*.

Suppose Robin Hood can shift to a higher-cost warfare. We raise δ so $2\delta > \tau_r > \tau_{lj}$. Now the mixed strategy solution obtains, and the payoff to Robin Hood is $(-\delta - \tau_{lj}/2)(\in + \tau_{lj}/2)/(\in + \delta)$, which it is easy to see is always less than $-\tau_{lj}$. Thus, *Robin Hood never wants to shift to a higher-cost form of fighting*, even though he would win some of the time.

6.17 The Motorist's Dilemma

Write $\sigma = \tau/2\delta < 1$, and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ represent George and Martha's mixed strategies, where (u_1, u_2) means play G with probability u_1, play W with probability u_2, and play C with probability $1 - u_1 - u_2$. Similarly for (v_1, v_2). Let $\delta = \{(x, y)|0 \le x, y, x + y \le 1\}$, so δ is the strategy space for both players.[1] It is easy to check that the payoff to the pair of mixed strategies (u, v) for George is

$$\begin{aligned} f_1(u, v) = -(2\delta v_1 + (\delta + \tau/2)(v_2 - 1))u_1 - ((\delta - \tau/2)(v_1 - 1) \\ + (\delta + \epsilon)v_2)u_2 + (\delta - \tau/2)v_1 \\ + (\delta + \tau/2)(v_2 - 1), \end{aligned} \tag{A6.1}$$

and the payoff $f_2(u, v)$ to Martha is, by symmetry, $f_2(u, v) = f_1(v, u)$. The players reaction sets are given by

$$R_1 = \{(u, v) \in \delta \times \delta | f_1(u, v) = \max_{\mu} f_1(\mu, v)\}$$
$$R_2 = \{(u, v) \in \delta \times \delta | f_2(u, v) = \max_{\mu} f_2(\mu, v)\},$$

and the set of Nash equilibria is $R_1 \cap R_2$.

If the coefficients of u_1 and u_2 are negative in equation (A6.1), then (0,0) is the only best response for George.

6.18 Family Politics

6.19 Frankie and Johnny

Let π be the payoff to Johnny, and write $\overline{x} = (x_f + x_j)/2$. If $x_f < x_j$, then $y < \overline{x}$ implies $\pi = x_f$, and otherwise $\pi = x_j$. If $x_f > x_j$, then $y < \overline{x}$ implies $\pi = x_j$, and otherwise $\pi = x_f$. Since $\Pr\{y < \overline{x}\} = F(\overline{x})$, we have $\pi = x_f F(\overline{x}) + x_j(1 - F(\overline{x}))$ for $x_f \le x_j$, and $\pi = x_j F(\overline{x}) + x_f(1 - F(\overline{x}))$ for $x_f > x_j$.

First, suppose $x_f < x_j$. The first-order conditions on x_f and x_j are then $\pi_{x_f} = F(\overline{x}) + f(\overline{x})(x_f - x_j)/2 = 0$, and $\pi_{x_j} = 1 - F(\overline{x}) + f(\overline{x})(x_f - x_j)/2 = 0$, from which it follows that $F(\overline{x}) = 1/2$. Substituting into the first order conditions gives

[1] As a notational convention, we write $\{x|p(x)\}$ to mean "the set of all x such that the assertion $p(x)$ is true."

$x_f = \overline{x} - 1/2f(\overline{x}), x_j = \overline{x} + 1/2f(\overline{x})$. Since π should be a minimum for Frankie, the second order condition must satisfy $\pi_{x_f x_f} = f(\overline{x}) + f'(\overline{x})(x_j - x_f)/4 > 0$. Since π should be a maximum for Johnny, the second order condition must satisfy $\pi_{x_j x_j} = -f(\overline{x}) + f'(\overline{x})(x_j - x_f)/4 < 0$.

For instance, if y is drawn from a uniform distribution then $\overline{x} = 1/2$ and $f(\overline{x}) = 1$, so $x_f = 0$ and $x_j = 1$. For another example, suppose $f(x)$ is quadratic, symmetric about $x = 1/2$, and $f(0) = f(1) = 0$. Then it is easy to check that $f(x) = 6x(1-x)$. In this case $\overline{x} = 1/2$ and $f(\overline{x}) = 3/2$, so $x_f = 1/6$ and $x_j = 5/6$.

6.20 A Card Game

The only undominated strategy for each player is to choose a critical level x_i^* and to fold if $x_i < x_i^*$. Let (x_1^*, x_2^*) be Nash strategies. The payoff to player 1 is

$$\begin{aligned} -1 \cdot \mathrm{P}[x_1 < x_1^*] + 1 \cdot \mathrm{P}[x_1 > x_1^*, x_2 < x_2^*] \\ - 6 \cdot \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] \\ + 6 \cdot \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1]. \end{aligned}$$

Clearly, we have

$$\mathrm{P}[x_1 < x_1^*] = x_1^*, \qquad \mathrm{P}[x_1 > x_1^*, x_2 < x_2^*] = (1 - x_1^*)x_2^*.$$

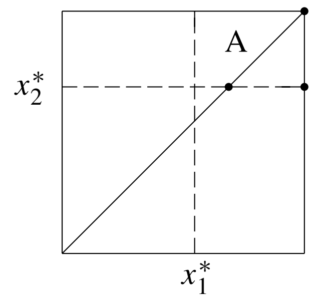
We also know

$$\begin{aligned} \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] + \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1] \\ = \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*] \\ = (1 - x_1^*)(1 - x_2^*). \end{aligned}$$

To evaluate $\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$, suppose $x_1^* > x_2^*$. Then,

$$\mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] = \mathrm{P}[x_1 > x_1^*, x_2 > x_1] = \frac{(1 - x_1^*)^2}{2}$$

To see this, consider the following diagram:

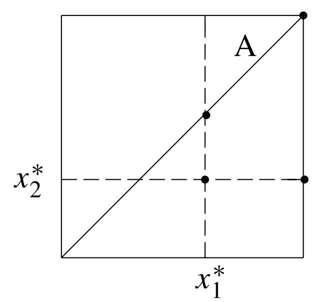


Since x_1 and x_2 are independently distributed, the pair (x_1, x_2) is uniformly distributed in the unit square depicted above. The case $P[x_1 > x_1^*, x_2 > x_1]$ is the little triangle labeled "A", which has area $(1 - x_1^*)^2/2$.

We thus have

$$P[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1] = (1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_1^*)^2}{2}.$$

To evaluate $P[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$ when $x_1^* < x_2^*$, refer to the following diagram:



Calculating the area of trapezoid A representing the case $P[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$, we get

$$P[x_1 > x_1^*, x_2 > x_2^*, x_1 < x_2] = (1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_2^*)^2}{2}.$$

Suppose $x_1^* > x_2^*$. The payoff to player 1 is then

$$\begin{aligned}\pi &= -x_1^* + (1 - x_1^*)x_2^* - 6\frac{(1 - x_1^*)^2}{2} \\ &\quad +6\left[(1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_1^*)^2}{2}\right] \\ &= 5x_1^* - 5x_2^* - 6x_1^{*2} + 5x_1^* x_2^*.\end{aligned}$$

The first-order condition on x_2^* is then $-5 + 5x_1^* = 0$, so $x_1^* = 1$. The first-order condition on x_1^* is $5 - 12x_1^* + 5x_2^* = 0$, so $x_2^* = 7/5$, which is impossible.

Thus, we must have $x_1^* < x_2^*$. The payoff to player 1 is then

$$-x_1^* + (1 - x_1^*)x_2^* - 6\left[(1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_2^*)^2}{2}\right] + 6\frac{(1 - x_2^*)^2}{2},$$

which reduces to

$$5x_1^* - 5x_2^* - 7x_1^*x_2^* + 6x_2^{*2}.$$

The first-order condition on x_1^* gives $x_2^* = 5/7$, and the first-order condition on x_2^* then gives $x_1^* = 25/49$. Note that we indeed have $x_1^* < x_2^*$. The payoff of the game to player 1 is then

$$5\frac{25}{49} - 5\frac{5}{7} + 6\left(\frac{5}{7}\right)^2 - 7\left(\frac{25}{49}\right)\left(\frac{5}{7}\right) = -\frac{25}{49}.$$

6.21 Cheater-Inspector

Let α be the probability of trusting. If there is a mixed strategy equilibrium in the n-round game, the payoff to cheating in the first period is $\alpha n + (1 - \alpha)(-an) = \alpha n(1 + a) - an$, and the payoff to being honest is $g_{n-1} + b(1 - \alpha)$. Equating these, we find

$$\alpha = \frac{g_{n-1} + b + an}{n(1 + a) + b},$$

assuming $g_{n-1} < n$ (which is true for $n = 0$, and which we will show is true for larger n by induction). The payoff of the n-round game is then

$$g_n = g_{n-1} + b\frac{n - g_{n-1}}{n(1 + a) + b}.$$

It is easy to check that $g_1 = b/(1 + a + b)$ and $g_2 = 2b/(1 + a + b)$, which suggests that

$$g_n = \frac{nb}{1 + a + b}.$$

This can be checked directly by assuming it to be true for g_{n-1} and proving it true for g_n. This is called “proof by induction:” prove it for $n = 1$, then show that it is true for some integer n, it is true for $n+1$. Then it is true for all integers n

$$\begin{aligned}
g_n &= g_{n-1} + b\frac{n-g_{n-1}}{n(1+a)+b} \\
&= \frac{b(n-1)}{1+a+b} + b\frac{n-\frac{b(n-1)}{1+a+b}}{n(1+a)+b} \\
&= \frac{b(n-1)}{1+a+b} + \frac{b}{1+a+b}\frac{n+na+nb-b(n-1)}{n(1+a)+b} \\
&= \frac{b(n-1)}{1+a+b} + \frac{b}{1+a+b} \\
&= \frac{bn}{1+a+b}.
\end{aligned}$$

6.22 The Vindication of the Hawk

6.23 Characterizing Two-player Symmetric Games I

6.24 Big Monkey and Little Monkey Revisited

Let σ be the mixed strategy for Big Monkey, who climbs with probability α, and let τ be the strategy for Little Monkey, who climbs with probability β. Let π_{c_i} and π_{w_i} be the payoffs to climbing and waiting, respectively, for player i. Then we have

$$\begin{aligned}
\sigma &= \alpha c_1 + (1-\alpha)w_1 \\
\tau &= \beta c_2 + (1-\beta)w_2 \\
\pi_{c_1} &= \beta\pi_1(c_1,c_2) + (1-\beta)\pi_1(c_1,w_2) \\
&\quad \text{where } \pi_1(c_1,c_2) = \text{BM's payoff to } c_1, c_2 \\
&= 5\beta + 4(1-\beta) = 4+\beta \\
\pi_{w_1} &= \beta\pi_1(w_1,c_2) + (1-\beta)\pi_1(w_1,w_2) \\
&= 9\beta + 0(1-\beta) = 9\beta \\
4+\beta &= 9\beta \Rightarrow \boxed{\beta = 1/2} \\
\pi_{c_2} &= \alpha\pi_2(c_1,c_2) + (1-\alpha)\pi_2(w_1,c_2)
\end{aligned}$$

$$\begin{aligned}
&= 3\alpha + (1-\alpha) = 1 + 2\alpha \\
\pi_{w_2} &= \alpha\pi_2(c_1, w_2) + (1-\alpha)\pi_2(w_1, w_2) \\
&= 4\alpha + 0(1-\alpha) = 4\alpha \\
1 + 2\alpha = 4\alpha &\Rightarrow \boxed{\alpha = 1/2}
\end{aligned}$$

Payoffs to Players:

$$\pi_1 = 5 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = \frac{9}{2}$$

$$\pi_2 = 3 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 2$$

Note: Show two other ways to find payoffs

6.25 Dominance Revisited

6.26 Competition on Main Street Revisited

6.27 Twin Sisters Revisited

6.28 Twin Sisters Simulated

6.29 One-Card Two-Round Poker with Bluffing

The normal form is as follows:

	ss	sf	f
rrbb	0,0	4,−4	2,−2
rrbf	1,−1	0,0	2,−2
rrf	2,−2	1,−1	0,0
rfbb	−5,5	0,0	2,−2
rfbf	−4,4	4,−4	2,−2
rff	−3,3	−3,3	0,0
fbb	−4,4	1,−1	0,0
fbf	−3,3	−3,3	0,0
ff	−2,2	−2,2	−2,2

The last six strategies for player 1 are weakly dominated by rrbb. Eliminating these strategies gives the following smaller normal form.

	ss	sf	f
rrbb	0,0	4,−4	2,−2
rrbf	1,−1	0,0	2,−2
rrf	2,−2	1,−1	0,0

If 2 uses α ss + β sf + $(1-\alpha-\beta)$ f, the payoffs to 1's strategies are:

$$\begin{aligned}\text{rrbb:}\quad & 4\beta + 2(1-\alpha-\beta) = -2\alpha + 2\beta + 2\\ \text{rrbf:}\quad & \alpha + 2(1-\alpha-\beta) = -\alpha - 2\beta + 2\\ \text{rrf:}\quad & 2\alpha + \beta\end{aligned}$$

If rrbb and rrbf are used, we have $\beta = \alpha/4$; if rrbb and rrf are used, we have $4\alpha = \beta + 2$. If rrbf and rrf are used we have $\alpha + \beta = 2/3$. Thus, if all three are used, we have $\alpha = 8/15$, $\beta = 2/15$, and $1-\alpha-\beta = 1/3$. The payoff is 18/15 = 6/5.

If 1 uses γ rrbb + δ rrbf + $(1-\gamma-\delta)$ rrf, the payoffs to 2's strategies are

$$\begin{aligned}\text{ss:}\quad & -\delta - 2(1-\gamma-\delta) = 2\gamma + \delta - 2\\ \text{sf:}\quad & -4\gamma - (1-\gamma-\delta) = -3\gamma + \delta - 1\\ \text{f:}\quad & -2\gamma - 2\delta.\end{aligned}$$

Thus, if ss and sf are used, $\gamma = 1/5$. If ss and f are both used, $4\gamma + 3\delta = 2$, so if all are used, $3\delta = 2 - 4/5 = 6/5$, and $\delta = 2/5$. Then $1-\gamma-\delta = 2/5$. The payoff is $4/5 - 2 = -1/5 - 1 = -6/5$, so it all works out.

There is a Nash equilibrium

$$\frac{8}{15}\text{ss} + \frac{2}{15}\text{sf} + \frac{1}{3}\text{f},$$
$$\frac{1}{5}\text{rrbb} + \frac{2}{5}\text{rrbf} + \frac{2}{5}\text{rrf},$$

with a payoff of 6/5 to player 1.

Note that we have arrived at this solution by eliminating weakly dominated strategies. Have we eliminated any Nash equilibria this way?

6.29.1 Simulating One-Card Two-Round Poker with Bluffing

6.30 Trust in Networks

You can check that $\pi^*(p) = 4(1-2p)^2/(1+p)(3p-1)$, which has derivative $8(1-7p+10p^2)/(1-3p)^2(1+p)^2$, which is positive for $p \in (0.5, 1]$.

6.31 Behavioral Strategies in Extensive Form Games

6.32 El Farol

For the second part, let p be the probability of going to the bar for each resident. The payoff to not going to the bar and the payoff to going to the bar must be equal. Thus, the payoff to going to the bar must be zero. This already proves that the social surplus must be zero. To find p, note that the probability that the other two people go to the bar is p^2, so the expected payoff to going to the bar is

$$2(1-p^2)+\frac{1}{2}p^2-1=0,$$

the solution to which is $p=\sqrt{6}/3$.

For the third part, let p_i be the probability of player i going to the bar, for $i=1,\ldots,3$. In a mixed strategy equilibrium, the payoff for each player to going to the bar and staying home must be the same. It is easy to show that this is equivalent to

$$p_i p_j = \frac{a_k - c_k}{a_k - b_k} \qquad i \neq j \neq k \neq i.$$

Let α_k be the right hand side of the above equation. We can solve the resulting three equations, getting $p_i = \sqrt{\alpha_j\alpha_k/\alpha_i}$. The conditions for a mixed strategy equilibrium are thus $\alpha_i\alpha_j < \alpha_k$ for $i \neq j \neq k \neq i$: if the costs and benefits of the bar are not too dissimilar for the three players, the mixed strategy equilibrium exists. Otherwise, one resident must always stay home.

6.33 Orange-Throat, Blue-Throat, and Yellow-Striped Lizards

6.34 Sex Ratios as Nash Equilibria

a. We have

$$\alpha = \frac{d}{s} = \frac{\sigma_f(1-v)}{\sigma_m v} \tag{A6.2}$$

and

$$f(u,v) = \sigma_f(1-u)c^2 + \sigma_m u c^2 \alpha \tag{A6.3}$$

To understand this expression, note that $\sigma_f(1-u)c$ is the number of daughters who survive to maturity, and so $\sigma_f(1-u)c^2$ is the number of grandchildren born to daughters. Similarly, $\sigma_m uc$ is the number of sons, and $\sigma_m uc(c\alpha)$ is the number of grandchildren sired by sons.

Substituting equation (A6.2) into equation (A6.3) and simplifying, we get

$$f(u,v) = c^2\sigma_f\left\{1 + u\left(\frac{1-2v}{v}\right)\right\}.$$

Thus, if $v \neq 1/2$, there cannot be a mixed strategy equilibrium: if the fraction of males in the population is less than 50%, each female should produce all males (i.e., set $u = 1$), and if the fraction of males in the population is greater than 50%, each female should produce all females (i.e., set $u = 0$). The only possible Nash strategy is therefore $u = v = 1/2$, since such a strategy must be symmetric (the same for all agents) and mixed (since all pure strategies are clearly not Nash).

b. Now suppose there are n females. It is easy to check that (A6.2) becomes

$$\alpha = \frac{d}{s} = \frac{\sigma_f[n-u-(n-1)v]}{\sigma_m[(n-1)v+u]}.$$

The number of grandchildren (A6.3) then becomes

$$f(u,v) = \sigma_f(1-u)c^2 + \sigma_m u c^2 \frac{\sigma_f[n-u-(n-1)v]}{\sigma_m[(n-1)v+u]}$$

$$= \frac{c^2\sigma_f}{(n-1)v+u}\{-2u^2 - u[2(n-1)v-(n+1)] + (n-1)v\}.$$

The first-order condition on u for maximizing $f(u,v)$ then gives

$$2(n-1)v = n+1-4u.$$

In a symmetric equilibrium, we must have $u = v$, which implies $u = v = 1/2$.

c. Now suppose a fraction δ_m of males and δ_f of females die in each period, and the rest remain in the mating pool. Let m be the number of males and let n be the number of females in the first period. Then the ratio α of females to males in the breeding pool in the next period is given by

$$\alpha = \frac{d+n(1-\delta_f)}{s+m(1-\delta_m)} = \frac{\sigma_f cn(1-v)+n(1-\delta_f)}{\sigma_m cnv+m(1-\delta_m)}. \tag{A6.4}$$

The number of grandchildren of one female who has fraction u of males and $1-u$ of females, when the corresponding fraction for other breeding females is v, is given by

$$f(u,v) = c^2\left[\sigma_f(1-u)+\sigma_m u\alpha\right] = c^2\left\{1+u\left[\frac{\sigma_m}{\sigma_f}\alpha - 1\right]\right\}.$$

Hence, a mixed strategy Nash equilibrium requires

$$\alpha = \frac{\sigma_f}{\sigma_m}. \tag{A6.5}$$

Solving (A6.4) and (A6.5) and simplifying, we get

$$v = \frac{1}{2}\left[1 - \frac{\sigma_f \gamma(1-\delta_m) - \sigma_m(1-\delta_f)}{\sigma_m \sigma_f c}\right], \tag{A6.6}$$

where we have written $\gamma = m/n$. But in the second period m is simply the denominator of (A6.4) and n is the numerator of (A6.4), so (A6.5) implies $\gamma = m/n = \sigma_m/\sigma_f$. Substituting this expression for γ in (6.6), we get

$$v = \frac{1}{2}\left[1 - \frac{\delta_f - \delta_m}{\sigma_f c}\right],$$

from which the result follows.

d. Now suppose males are haploid and females are diploid. Then, for a female who has fraction u of sons and $1-u$ of daughters, when the corresponding fraction for other breeding females is v, the fraction of genes in daughters is $c(1-u)/2$, and the fraction in sons is cu. The number of genes (normalizing the mother's gene complement to unity) in daughters of daughters is $c^2(1-u)(1-v)/4$, the number of genes in sons of daughters is $c^2(1-u)v/2$, and the number of genes in daughters of sons is $c^2 u\alpha(1-v)$. None of the female's genes are in sons of sons, since only the mother passes genetic material to her sons. The number of genes in the mother's grandchildren is the sum of these three components, which simplifies to

$$f(u,v) = c^2\left\{\frac{1+v}{4} - u\left[\frac{1+v}{4} - (1-v)\alpha\right]\right\},$$

so we must have

$$\alpha = \frac{1+v}{4(1-v)}. \tag{A6.7}$$

But by our assumption that individuals live for only one breeding period, (A6.2) still holds. Solving (A6.2) and (A6.7) simultaneously, and defining $\nu = \sigma_f/\sigma_m$, we get

$$v = \frac{1 + 8\nu \pm \sqrt{32\nu_{+}1}2(4\nu - 1)}{},$$

where the sign of the square root is chosen to ensure $0 < v < 1$. This implies that, for instance, if $\sigma_f = \sigma_m$, then $v \approx 0.54$; i.e., the ratio of daughters to sons should be only slightly biased towards males.

6.35 A Mating Game

A mixed strategy for a female is a pair of probabilities (α_H, α_E), where α_H is the probability of being forward when she is H, and α_E is the probability of being forward when she is E. A mixed strategy for a male is a pair of probabilities (β_H, β_E), where β_H is the probability of being forward when he is H, and β_E is the probability of being forward when he is E. Let $\alpha = \alpha_H + \alpha_E$, and $\beta = \beta_H + \beta_E$. You can check that the payoffs for males are (a) $\pi^m_{FF} = 1$; (b) $\pi^m_{FR} = 3(2-\alpha)/4$; (c) $\pi^m_{RF} = 3\alpha/4$; (d) $\pi^m_{RR} = 1$. The payoffs for females are (a) $\pi^f_{FF} = 1 - \beta/4$; (b) $\pi^f_{FR} = 2 - \beta$; (c) $\pi^f_{RF} = \beta/2$; (d) $\pi^f_{RR} = 1 - \beta/4$. Also, $\alpha, \beta = 2$ for FF, $\alpha, \beta = 1$ for FR and RF, and $\alpha, \beta = 0$ for RR. Now you can form the 4×4 normal form matrix, and the rest is straightforward.

6.36 Coordination Failure

We'll do the second part. Suppose 1 uses the three pure strategies with probabilities α,β, and $\gamma = 1-\alpha-\beta$, respectively, and 2 uses the pure strategies with probabilities a, b, and $c = 1 - a - b$. We can assume without loss of generality that $\alpha > 1/3$ and $\beta \geq \gamma$. The payoffs to a, b, and c are

$$\begin{aligned}
\pi_a &= 50\beta + 40(1-\alpha-\beta) = 40 - 40\alpha + 10\beta,\\
\pi_b &= 40\alpha + 50(1-\alpha-\beta) = 50 - 10\alpha - 50\beta,\\
\pi_c &= 50\alpha + 40\beta.
\end{aligned}$$

We have $\alpha + 2\beta \geq 1$, so $\beta \geq (1-\alpha)/2$. Then,

$$\begin{aligned}
\pi_c - \pi_a &= 50\alpha + 40\beta - [40 - 40\alpha + 10\beta] = 90\alpha + 30\beta - 40\\
&> 90\alpha + 30(1-\alpha)/2 - 40 = 15 + 75\alpha - 40 = 75\alpha - 25 > 0.
\end{aligned}$$

Thus, c is better than a. Also,

$$\begin{aligned}
\pi_c - \pi_b &= 50\alpha + 40\beta - [50 - 10\alpha - 50\beta] = 60\alpha + 90\beta - 50\\
&> 60\alpha + 90(1-\alpha)/2 - 50 = 45 + 15\alpha - 50 = 15\alpha - 5 > 0,
\end{aligned}$$

so c is better than b. Thus, player 2 will use c, and his payoff is $50\alpha + 40\beta > 50\alpha + 20(1-\alpha) = 20 + 30\alpha > 30$. The payoff to 1 is then $40\alpha + 50\beta > 40\alpha + 25(1-\alpha) = 25 + 15\alpha > 30$. Thus, both are better off than with the 30 payoff of the Nash equilibrium.

6.37 Colonel Blotto Game

The payoff matrix, giving Colonel Blotto's return (the enemy's payoff is the negative of this) is as follows:

		Enemy Strategies			
		(3,0)	(0,3)	(2,1)	(1,2)
	(4,0)	4	0	2	1
Colonel	(0,4)	0	4	1	2
Blotto	(3,1)	1	−1	3	0
Strategies	(1,3)	−1	1	0	3
	(2,2)	−2	−2	2	2

Suppose the enemy uses all strategies. By symmetry, 1 and 2 must be used equally, and 3 and 4 must be used equally. Let p be the probability of using (3,0), and q be the probability of using (2,1). The expected return to Colonel Blotto is then

$$\begin{array}{rrrrcl} 4p+ & & 2q+ & q & = & 4p+3q\\ & 4p+ & q & +2q & = & 4p+3q \\ p- & p & +3q & & = & 3q \\ -p+ & p & & +3q & = & 3q\\ -2p & -2p & +2q & +2q & = & -4p+4q. \end{array}$$

Colonel Blotto cannot use all strategies in a mixed strategy, since there is no p that makes all entries in this vector equal. Suppose we drop Colonel Blotto's (3,1) and (1,3) strategies and choose p to solve $4p + 3q = -4p + 4q$ and $2p + 2q = 1$. Thus, $p = 1/18$ and $q = 4/9$. There are other Nash equilibria.

6.38 Number Guessing Game

Clearly, the game is determined in the first two rounds. Let us write my strategies as (g h l), for "first guess g, if high guess h and if low guess l." If a high guess is impossible, we write (1 0 l), and if a low guess is impossible, we write (3 h 0). For instance, (103) means ”first choose 1, and if this is low, then choose 3.” Then, we have the following payoff matrix for player 1 (the payoff to player 2 is minus the payoff to player 1),

My Choices

Your Choices		(102)	(103)	(213)	(310)	(320)
	1	1	1	2	2	3
	2	2	3	1	3	2
	3	3	2	2	1	1

Suppose you choose 1, 2 and 3 with probabilities α, β, and $1-\alpha-\beta$. Then, I face $[-2\alpha-\beta+3, -\alpha+\beta+2, 2-\beta, \alpha+2\beta+1, 2\alpha+\beta+1]$. Clearly, I can't use all actions. Suppose I drop (102) and (320). Then, equating the costs of the other three, I get $\alpha=2/5$ and $\beta=1/5$, with cost 9/5. The costs of (102) and (320) are $-2/5-2/5+3=2>9/5$. Therefore, my choices can be part of a Nash equilibrium. Suppose I choose (103), (213), and (310) with probabilities p, q and $1-p-q$. If you use all three numbers, your payoffs are $[-p+2, -2q+3, p+q+1]$. These are equal when $p=1/5$ and $q=3/5$, with payoff 9/5.

6.39 Target Selection

a. Suppose attacker uses mixed strategy $x=(x_1,\ldots,x_n)$ and defender uses strategy $y=(y_1,\ldots,y_n)$, and these form a Nash equilibrium. If $x_j=0$, then the best response of defender must set $y_j=0$. Suppose $x_i>0$ for some $i>j$. Then, by switching x_i and x_j, attacker gains $a_j-pa_iy_i\geq a_j-a_i>0$.
b. All payoffs to pure strategies used with positive probability in a best response must be equal when played against the mixed strategies of the other player(s).

6.40 A Reconnaissance Game

The normal form matrix is as follows:

	counter full defend	counter half defend	no counter full defend	no counter half defend
reconnoiter, full attack	$a_{11}-c+d$	$a_{12}-c+d$	$a_{11}-c$	$a_{22}-c$
reconnoiter, half attack	$a_{21}-c+d$	$a_{22}-c+d$	$a_{11}-c$	$a_{22}-c$
no reconnoiter, full attack	$a_{11}+d$	$a_{12}+d$	a_{11}	a_{12}
no reconnoiter, half attack	$a_{21}+d$	$a_{22}+d$	a_{21}	a_{22}

With the given payoffs and costs, the entries in the normal form game become

46,−46	22,−22	39,−39	27,−27
10,−10	34,−34	39,−39	27,−27
55,−55	31,−31	48,−48	24,−24
19,−19	43,−43	12,−12	36,−36.

Suppose defender doesn't counter and full defends with probability p. Then, attacker faces

$$
\begin{aligned}
39p + 27(1-p) &= 12p + 27 \\
39p + 27(1-p) &= 12p + 27 \\
48p + 24(1-p) &= 24p + 24 \\
12p + 36(1-p) &= -24p + 36.
\end{aligned}
$$

Check the third and fourth. We have $-24p + 36 = 24p + 24$, so $p = 1/4$. Suppose attacker doesn't reconnoiter and full attacks with probability q. Then, $-48q - 12(1-q) = -24q - 36(1-q)$, so $q = 1/2$. You must check that no other strategy has a higher payoff, and you will find this to be true. The payoffs are $(30, -30)$. If you are ambitious, you can check that there are many other Nash equilibria, all of which involve (0,0,1/4,3/4) for Player 2. How do you interpret this fact?

6.41 Attack on Hidden Object

We have

	P	F
pp	$2\gamma - \gamma^2$	$\beta\gamma$
pF	γ	β
Fp	$\beta^2 - \beta\gamma + \gamma$	β
FF	β^2	$2\beta - \beta^2$

Note that the second row is weakly dominated by the third. Let p be the probability of defender using P. Then, attacker faces

$$p(2\gamma - \gamma^2) + (1-p)\beta\gamma = p\gamma(2-\gamma-\beta) + \beta\gamma$$
$$p(\beta^2 - \beta\gamma + \gamma) + (1-p)\beta = p(\beta^2 - \gamma\beta - \beta + \gamma) + \beta$$
$$p\beta^2 + (1-p)(2\beta - \beta^2) = -2p\beta(1-\beta) + \beta(2-\beta).$$

If attacker uses PP and PF, then

$$p = \frac{\beta(1-\gamma)}{\gamma(1-\gamma) + \beta(1-\beta)}.$$

But if attacker uses PP with probability q and PF with probability $(1-q)$, then defender faces

$$[q(2\gamma - \gamma^2) + (1-q)\gamma q\beta\gamma + (1-q)\beta] = [q\gamma(1-\gamma) + \gamma - q\beta(1-\gamma) + \beta],$$

so

$$q = \frac{\beta - \gamma}{(\gamma+\beta)(1-\gamma)}.$$

Try to find some other equilibria.

6.42 Domination by Mixed Strategies

6.43 Two-Person Zero-Sum Games

6.44 An Introduction to Forward Induction

6.45 Mutual Monitoring in a Partnership

6.46 Mutual Monitoring in Teams

6.47 Altruism(?) in Bird Flocks

6.48 The Groucho Marx Game

a. Staying if you get the 3 is strictly dominated by raising, so there are four strategies left: rr, rs, sr,and ss (refer to the hint in the question for the meaning of these strategies). Here are the payoffs, where down the first column, 12 means (player 1 draws 1, player 2 draws 2), etc.

	rr/rr	rr/rs	rr/sr	rr/ss	rs/rs	rs/sr	rs/ss	sr/sr	sr/ss	ss/ss
12	$-a-b$	a	$-a-b$	a	a	$-a-b$	a	$-a$	0	0
13	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a$	$-a$	$-a$
21	$a+b$	$a+b$	a	a	$-a$	0	0	a	a	0
23	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a$	$-a$	$-a$	$-a-b$	$-a-b$	$-a$
31	$a+b$	$a+b$	a	a	$a+b$	a	a	a	a	a
32	$a+b$	a	$a+b$	a	a	$a+b$	a	$a+b$	a	a
	0	$2a$	$-2b$	$2(a-b)$	0	$-a-b$	$a-b$	0	$a-b$	0

The answer follows directly from the resulting payoff matrix:

	rr	rs	sr	ss
rr	0	$a/3$	$-b/3$	$(a-b)/3$
rs	$-a/3$		$-(a+b)/3$	$(a-b)/3$
sr	$b/3$	$(a+b)/3$	0	$(a-b)/3$
ss	$-(a-b)/3$	$-(a-b)/3$	$-(a-b)/3$	0

b. Staying when you pick the 4 is strictly dominated by raising. This leaves us with eight strategies for each player. Note that staying with 2 and raising with 1 is weakly dominated by staying with 1 or 2 (explain). This generalizes to the conclusion that you only eliminate dominated strategies by staying unless the card you pick is greater than some number between zero and three. Thus, four strategies remain: {rrr, srr, ssr, sss}. The payoff of any strategy against itself is clearly zero. Thus, it remains to calculate the following:

	rrr/srr	rrr/ssr	rrr/sss	srr/ssr	srr/sss	ssr/sss
12	$-a-b$	a	a	0	0	0
13	$-a-b$	$-a-b$	a	$-a$	0	0
14	$-a-b$	$-a-b$	$-a-b$	$-a$	$-a$	$-a$
21	a	a	a	a	a	0
23	$-a-b$	$-a-b$	a	$-a-b$	a	0
24	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a$
31	a	a	a	a	a	a
32	$a+b$	a	a	a	a	a
34	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$
41	a	a	a	a	a	a
42	$a+b$	a	a	a	a	a
43	$a+b$	$a+b$	a	$a+b$	a	$a+b$
	$-3b$	$2a-4b$	$6a-3b$	$a-2b$	$4a-2b$	$2a$

Here is twelve times the payoff matrix (only player 1's payoffs listed), from which the result follows:

	rrr	srr	ssr	sss
rrr	0	$-3b$	$2a-4b$	$6a-3b$
srr	$3b$	0	$a-2b$	$4a-2b$
ssr	$-2a+4b$	$-a+2b$	0	$2a$
sss	$6a-3b$	$-4a+2b$	$-2a$	0

c. We represent the strategy of raising if and only if the card chosen is greater than k by s_k. Thus, each player has n pure strategies (eliminating weakly dominated strategies). We must find the payoff to each pure strategy pair $\{(s_k, s_l)|k, l = 1, \ldots, n\}$. Suppose the pure strategies used are (s_k, s_l) and the cards picked from the hat by players 1 and 2 are $\tilde{k}$ and $\tilde{l}$, respectively.

First suppose $k \geq l$. The probability player 1 wins if both stay is

$$\begin{aligned}
\mathrm{P}\left[\tilde{k} > \tilde{l}|\tilde{k} \leq k, \tilde{l} \leq l\right] &= \mathrm{P}\left[\tilde{k} \leq l\right]\mathrm{P}\left[\tilde{k} > \tilde{l}|\tilde{k}, \tilde{l} \leq l\right] \\
&\quad + \mathrm{P}\left[\tilde{k} > l|\tilde{k} \leq k, \tilde{l} \leq l\right] \\
&= \frac{l}{k}\frac{1}{2} + \frac{k-l}{k} = 1 - \frac{l}{2k}.
\end{aligned}$$

Since the probability that player 1 loses if both stay is one minus the above quantity, and player 1 stands to win or lose a in this case, we find that player 1's expected payoff in this case is

$$\pi_{k \geq l}\left[\tilde{k} > \tilde{l}|\tilde{k} \leq k, \tilde{l} \leq l\right] = a\left(1 - \frac{l}{k}\right).$$

By symmetry (interchange k and l and then negate—or you can calculate it out), we have

$$\pi_{k<l}\left[\tilde{k} > \tilde{l}|\tilde{k} \leq k, \tilde{l} \leq l\right] = -a\left(1 - \frac{k}{l}\right).$$

We also have the following easy payoffs:

$$\pi\left[\tilde{k} > k, \tilde{l} \leq l\right] = a$$

$$\pi\left[\tilde{k} \leq k, \tilde{l} > l\right] = -a$$

Finally, suppose both players raise. First assume $k \geq l$. Then,

$$\mathrm{P}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] = \mathrm{P}\left[\tilde{k} > \tilde{l} | \tilde{k}, \tilde{l} > k\right] + \mathrm{P}\left[\tilde{l} \leq k | \tilde{l} > l\right]$$
$$= \frac{n-k}{n-l}\frac{1}{2} + \frac{k-l}{n-l}.$$

Since the probability that player 1 loses if both raise is one minus the above quantity, and player 1 stands to win or lose $a + b$ in this case, we find that player 1's expected payoff in this case is

$$\pi_{k \geq l}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] = (a+b)\frac{k-l}{n-l}.$$

By symmetry (or you can calculate it out), we have

$$\pi_{k < l}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] = (a+b)\frac{k-l}{n-k}.$$

Now we add everything up:

$$\begin{aligned}\pi_{k \geq l} = {} & \mathrm{P}\left[\tilde{k} \leq k\right]\mathrm{P}\left[\tilde{l} \leq l\right]\pi_{k \geq l}\left[\tilde{k} > \tilde{l} | \tilde{k} \leq k, \tilde{l} \leq l\right] \\ & + \mathrm{P}\left[\tilde{k} \leq k\right]\mathrm{P}\left[\tilde{l} > l\right]\pi\left[\tilde{k} \leq k, \tilde{l} > l\right] \\ & + \mathrm{P}\left[\tilde{k} > k\right]\mathrm{P}\left[\tilde{l} \leq l\right]\pi\left[\tilde{k} > k, \tilde{l} \leq l\right] \\ & + \mathrm{P}\left[\tilde{k} > k\right]\mathrm{P}\left[\tilde{l} > l\right]\pi_{k \geq l}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] \\ = {} & \frac{1}{n^2}(l-k)(a(k-l) - b(n-k)).\end{aligned}$$

By symmetry (or calculation if you don't trust your answer—I did it by calculation and checked it by symmetry), we have

$$\pi_{k < l} = \frac{1}{n^2}(k-l)(a(k-l) + b(n-l)).$$

6.49 Real Men Don't Eat Quiche

Let B (drink beer) and Q (eat quiche) be the actions for Clem, and B (bully Clem) and D (defer to Clem) be the actions for the Dark Stranger. The strategies for

Clem are BB (drink beer), BQ (drink beer if Tough, eat Quiche if Wimp), QB (eat quiche if Tough, drink beer if Wimp), and QQ (eat quiche), and those for the Dark Stranger are BB (bully), DB (defer if Clem drinks beer, bully if Clem eats quiche), BD (bully if Clem drinks beer, defer if Clem eats quiche), and DD (defer). The payoff matrices in the Tough and Wimp cases are given in the following diagram, where $a, b/c, d$ means "Clem's payoff is a if Tough and c if Wimp; Dark Stranger's payoff is b if Clem is Tough, and d if Clem is a Wimp":

	BB	BD	DB	DD
BB	1,0/0,1	1,0/0,1	3,1/2,0	3,1/2,0
BQ	1,0/1,1	1,0/3,0	3,1/1,1	3,1/3,0
QB	0,0/0,1	2,1/0,1	0,0/2,0	2,1/2,0
QQ	0,0/1,1	2,1/3,0	0,0/1,1	2,1/3,0

The expected return to the players is thus one-third times the first matrix plus two-thirds times the second matrix, since one-third is the probability that Clem is Tough. We get the payoff matrix (all payoffs multiplied by 3):

	BB	BD	DB	DD
BB	1,2	1,2	7,1	7,1
BQ	3,2	7,0	5,3	9,1
QB	0,2	2,3	4,0	6,1
QQ	2,2	8,1	2,2	8,1

Now QB is dominated by BQ, DD is dominated by BB, and BD is dominated by BB (since QB is out). But then QQ is dominated by BQ, so we have

	BB	DB
BB	1,2	7,1
BQ	3,2	5,3

This is sensible: Clem can bluff Tough or not, and the Dark Stranger can call the bluff or not. There is a mixed strategy equilibrium. Let α be the probability of

BB. The payoff to BB and DB must then be equal for the Dark Stranger, so $2 = \alpha + 3(1-\alpha)$, so $\alpha = 1/2$. If β is the probability of the Dark Stranger using BB, then $\beta + 7(1-\beta) = 3\beta + 5(1-\beta)$, or $\beta = 1/2$.

6.50 Games of Perfect Information

6.51 Correlated Equilibria

6.52 Poker with Bluffing Revisited

6.53 The Equivalence of Behavioral and Mixed Strategies

a. Suppose player i has information sets $\nu_1, \ldots, \nu_k$, so a pure strategy for i can be written as $a_1 a_2 \ldots a_k$, where $a_j \in \nu_j$ for $j = 1, \ldots, k$. Then,

$$\sum_{s \in S_i} \alpha_s^p = \sum_{a_1 \in \nu_1} \ldots \sum_{a_k \in \nu_k} p_i(a_1) \cdots p_i(a_k)$$

$$= \sum_{a_1 \in \nu_1} p_i(a_1) \ldots \sum_{a_k \in \nu_k} p_i(a_k) = 1.$$

b. Let σ be the mixed strategy representation of behavioral strategy p. We must show that for every player i and every information set $\nu \in \mathcal{N}_i$, if $\mathrm{P}[\nu|\sigma_i] > 0$, then for every $a \in \alpha^\nu$, we have $p(a) = \mathrm{P}[a|\sigma_i]/p[\nu|\sigma_i]$. If $N \subseteq \mathcal{N}_i$, we denote by S_i/N the set of pure strategies over the nodes $\{\nu|\nu \in \mathcal{N}_i - N\}$, where for sets A and B, $A - B = \{a \in A | a \notin B\}$; i.e., $s \in S_i/N$ is a choice of a node in each of player i's information sets not in N. Similarly, we denote by $S_i/N[a]$ the pure strategies over $\mathcal{N}_i - N$ that lead to node a for some choice of pure strategies of the other players, and we denote by $S_i/N[\nu]$ the pure strategies over $\mathcal{N}_i - N$ that lead to information set ν for choice of pure strategies of the other players. Finally, $n(\mu, a)$ is the statement "$\mu \in \mathcal{N}_i$ is not in the path from the root node r to node a." Then, for $\sigma_i = \sum_{s \in S_i} \alpha_s s$, $a \in \nu \in \mathcal{N}_i$, $\mu' \in N = \{\mu \in \mathcal{N}_i | n(\mu, a)\}$, we have

$$\mathrm{P}[a|\sigma_i] = \sum_{s \in S_i[a]} \alpha_s$$

$$= \sum_{s \in S_i[a]} \prod_{\mu \in \mathcal{N}_i} p_i(s_i(\mu))$$

$$= \sum_{s \in S_i[a]} p_i(s_i(\mu')) \prod_{\substack{\mu \in \mathcal{N}_i \\ \mu \neq \mu'}} p_i(s_i(\mu))$$

$$= \sum_{s \in S_i/\{\mu'\}[a]} \prod_{\substack{\mu \in \mathcal{N}_i \\ \mu \neq \mu'}} p_i(s_i(\mu))$$

$$= \sum_{s \in S_i/N[a]} \prod_{\substack{\mu \in \mathcal{N}_i \\ \mu \notin N}} p_i(s_i(\mu))$$

$$= \prod_{\substack{\mu \in \mathcal{N}_i \\ \mu \notin N}} p_i(s_i^a(\mu)),$$

where s_i^a is the unique member of S_i/N that chooses at each information set on the path from r to a, the branch belonging to that path. But the last expression is the product of the probabilities assigned by p to i's choices along the path from r to a. A similar argument shows that $P[\nu|\sigma_i]$ is the sum of the product of the probabilities assigned by p to i's choices along all paths from r to nodes in ν. But by the assumption of perfect recall, i must make the same choices to get to any node in ν, so there is only one such path. It follows that, provided the denominator is not zero, the behavioral probability assigned to node a is just

$$\frac{P[a|\sigma_i]}{P[\nu|\sigma_i]} = p_i(a),$$

which proves that p is a behavioral representation of σ.

c. The payoff to player i given by behavioral strategy p is

$$\pi_i(p) = \sum_{t \in T} P[r, t|p]\pi_i(t),$$

and the payoff given by mixed strategy σ is

$$\pi_i(\sigma) = \sum_{t \in T} P[r, t|\sigma]\pi_i(t),$$

where $P[r, a|\sigma]$ is the probability of reaching a from the root node r if all players randomize their choice of pure strategies according to the weighting in σ. To complete the proof, we must show that $P[r, t|\sigma] = P[r, t|p]$.[2] We will show that for every node a, $P[r, a|\sigma] = P[r, a|p]$. This is clearly true for $a = r$, since then both sides of the equation are unity. Suppose it is true for node a, and a' is a child node of a. If Nature moves at a and chooses a' with probability q, then clearly $P[r, a'|p] = qP[r, a|p]$. Let $p_N(a)$ be the product of the probabilities of all the branches associated with Nature on the path from r to a. Then, if $\sigma_i = \sum_{s\in S_i} \alpha_s s$, we have

$$
\begin{aligned}
P[r, a'|\sigma] &= p_N(a') \prod_{j=1}^{n} \sum_{s_j \in S_j[a']} \alpha_{s_j} \\
&= p_N(a) \prod_{j=1}^{n} \sum_{s_j \in S_j[a']} \alpha_{s_j} \\
&= p_N(a) \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \sum_{s_i \in S_i[a']} \right) \sum_{s_i \in S_i[a']} \alpha_{s_i} \\
&= p_N(a) \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \sum_{s_i \in S_i[a']} \right) q \sum_{s_i \in S_i[a]} \alpha_{s_i} \\
&= P[r, y|\sigma]q.
\end{aligned}
$$

Note that we used perfect recall in the next-to-last step, where we used the equation

$$\sum_{s_i \in S_i[a']} \alpha_{s_i} = q \sum_{s_i \in S_i[a]} \alpha_{s_i}.$$

Under perfect recall, $S_i[a] = S_i[\nu]$, where $a \in \nu$, and with this substitution, the equation is the definition of $q = p(a')$.

d. If p_ν is not a best response, then there is some q_ν that gives i a higher payoff. Then, the mixed strategy representation of $(q_\nu, p_{-\nu})$ gives i a higher payoff than σ_i, which contradicts the fact that σ_i is a best response.

[2]The reader will note that to avoid a proliferation of new symbols, I have been using what is known as "functional overloading," by which is meant that two functions with the same function name but with different arguments represent two distinct functions. Thus, $P[a|\sigma_i]$, $P[\nu|\sigma_i]$, $P[r, a|\sigma]$, and $P[r, a|p]$ are notationally unambiguous but have distinct meanings, so long as a is a node, ν is an information set, σ_i is a mixed strategy, and r is the root node of the game tree.

6.54 Harsanyi's Purification Theorem

6.55 An Epistemic Approach to Mixed Strategies in One-shot Games

7

Moving through the Game Tree: Subgames, Incredible Threats, and Trembling Hands

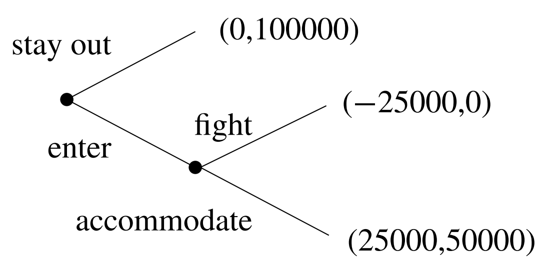
7.1 Introduction

7.2 Subgame Perfection

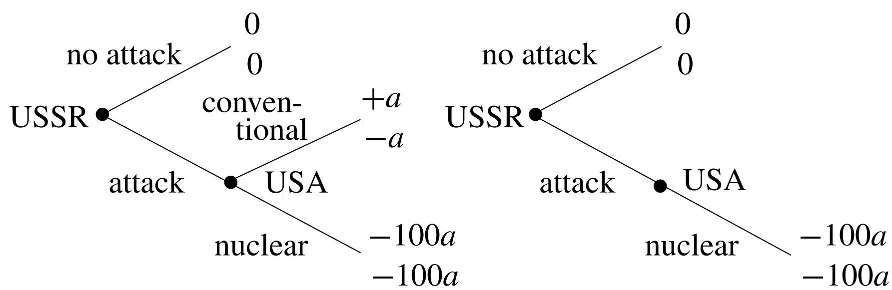
7.3 Stackelberg Leadership

7.4 The Subway Entry Deterrence Game

The game tree is as follows:



7.5 The Dr. Strangelove Game



USA Has No Credible Threat   USA Precommits to Nuclear Deterrent

a. The figure on the left is the extensive form of the game without precommitment.
b. The figure on the right is the extensive form of the game with precommitment.

7.6 The Rubinstein Bargaining Model

7.7 Rubinstein Bargaining with Heterogeneous Impatience

7.8 Rubinstein Bargaining with One Outside Option

Let $1-x$ be the maximum player 1 can get in any subgame perfect Nash equilibrium, assuming player 2 accepts the outside offer when it is available, and assuming $1-x \geq 0$. Then when it is player 2's turn to offer, he must offer at least $\delta(1-x)$, so his maximum payoff when rejecting x is $ps_2 + (1-p)\delta(1-\delta(1-x))$. The most player 1 must offer player 2 is this amount, so the most he can make satisfies the equation

$$1 - x = 1 - (ps_2 + (1-p)\delta(1-\delta(1-x))),$$

which gives

$$x = \frac{ps_2 + (1-p)\delta(1-\delta)}{1-(1-p)\delta^2}.$$

A similar argument shows that x is the minimum player 1 can get in a subgame perfect Nash equilibrium, assuming player 2 accepts the outside offer when it is available, and $1-x \geq 0$. This shows that such an x is unique. Our assumption that player 2 accepts the outside offer requires

$$s_2 \geq \delta(1-\delta(1-x)),$$

which is player 2's payoff if he rejects the outside option. It is easy to show that this inequality holds exactly when $s_2 \geq \delta/(1+\delta)$. We also must have $1-x \geq 0$, or player

1 will not offer x. It is easy to show that this is equivalent to $s_2 \leq (1-\delta(1-p))/p$. The rest of the argument is straightforward.

7.9 Rubinstein Bargaining with Dual Outside Options

Throughout the answer, we use the one-stage deviation principle (§8.6) without comment, where appropriate. The game tree is depicted in Fig. 7.1, where a means "accept" and r means "reject". Note that this is not a complete game tree, because we do not represent the players' decisions concerning accepting or rejecting the outside offer.

For Case 1, suppose $s_1, s_2 < \delta/(1+\delta)$. Then the spNe of the Rubinstein bargaining model without outside options is an spNe of this game, with the added proviso that neither player takes the outside option when it is available. To see that rejecting the outside option is a best response, note that when Two has the outside option, he also knows that if he rejects it, his payoff will be $\delta/(1+\delta)$, so he should reject it. A similar argument holds for One.

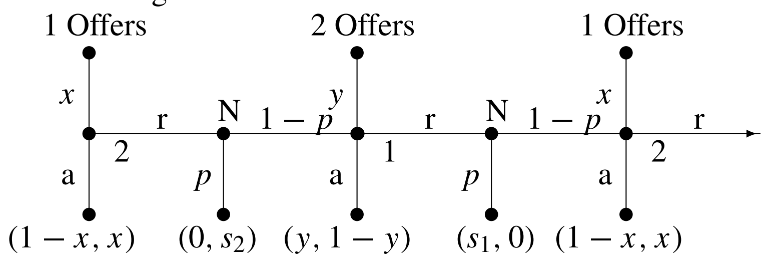


Figure 7.1. Rubinstein Bargaining with Dual Outside Options.

For the Case 2, assuming both agents take the outside option when it is available, show that we have the recursive equation

$$x = ps_2 + \delta(1-p)(1-(ps_1 + (1-p)\delta(1-x))),$$

and the above inequalities ensure that $x \in [0, 1]$, that $s_1 > \delta(1-x)$, so One takes the outside option when available, and $s_2 > \delta(1 - ps_1 - (1-p)\delta(1-x))$, so Two takes the outside option when available. This justifies our assumption.

For Case 3, first show that if One could make an acceptable offer to Two, then the previous recursion for x would hold, but now $x < 0$, which is a contradiction. Then either One accepts an offer from Two, or One waits for the outside option to become available. The payoff to waiting is

$$\pi_1 = \frac{(1-p)ps_1\delta}{1-(1-p)^2\delta^2},$$

but One will accept $\delta(ps_1+(1-p)\delta\pi_1)$, leaving Two with $1-\delta(ps_1+(1-p)\delta\pi_1)$. This must be better for Two than just waiting for the outside option to become available, which has value, at the time Two is the proposer,

$$\frac{(1-p)ps_2\delta}{1-(1-p)^2\delta^2}.$$

Show that the above inequalities imply Two will make an offer to One. Then use the inequalities to show that One will accept. The remainder of the problem is now straightforward.

For Case 4, One must offer Two

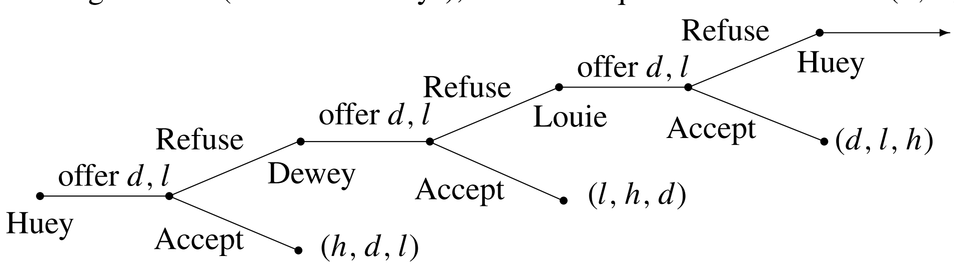
$$\pi_2 = \frac{ps_2}{1-(1-p)^2\delta^2},$$

which is what Two can get by waiting for the outside option. One minus this quantity must be greater than π_1, or One will not offer it. Show that this holds when the above inequalities hold. Now, One will accept $ps_1 + (1-p)\delta\pi_1$, but you can show that this is greater than 1, so Two will not offer this. This justifies our assumption that One will not accept anything that Two is willing to offer.

For Case 5, by refusing all offers and waiting for the outside option to become available, One's payoff is π_1 and Two's payoff is π_2. Show that the inequalities imply $1-\pi_2 < \pi_1$, so ONe will not make an acceptable offer to Two, and $1-\pi_1/\delta(1-p) < \delta(1-p)\pi_2$, so Two will not make an acceptable offer to One.

7.10 Huey, Dewey, and Louie Split a Dollar

Here is a game tree (written sideways), when the equilibrium shares are (h, d, l):



We work back the game tree (which is okay, since we are looking only for subgame perfect equilibria). At the second place where Huey gets to offer (at the right side of the game tree), the value of the game to Huey is h, since we assume a stationary equilibrium. Thus, Louie must offer Huey at least δh where Louie gets to offer, to get Huey to accept. Similarly, Louie must offer Dewey at least δd at this node. Thus, the value of the game where Louie gets to offer is $(1 - \delta h - \delta d)$.

When Dewey gets to offer, he must offer Louie at least δ times what Louie gets when it is Louie's turn to offer, to get Louie to accept. This amount is just $\delta(1 - \delta h - \delta d)$. Similarly, he must offer Huey $\delta^2 h$ to accept, since Huey gets δh when it is Louie's turn to offer. Thus, Dewey gets

$$1 - \delta(1 - \delta h - \delta d) - \delta^2 h = 1 - \delta(1 - \delta d)$$

when it is his turn to offer.

Now Huey, on his first turn to offer, must offer Dewey δ times what Dewey can get when it is Dewey's turn to offer, or $\delta(1 - \delta(1 - \delta d))$. But then we must have

$$d = \delta(1 - \delta(1 - \delta d)).$$

Solving this equation for d, we find

$$d = \frac{\delta}{1 + \delta + \delta^2}.$$

Moreover, Huey must offer Louie δ times what Dewey would offer Louie in the next period or $\delta^2(1 - \delta h - \delta d)$. Thus, Huey offers Dewey and Louie together

$$\delta(1 - \delta(1 - \delta d)) + \delta^2(1 - \delta h - \delta d) = \delta - \delta^3 h,$$

so Huey gets $1 - \delta + \delta^3 h$, and this must equal h. Solving, we get

$$h = \frac{1}{1 + \delta + \delta^2},$$

so we must have

$$l = 1 - d - h = \frac{\delta^2}{1 + \delta + \delta^2},$$

which is the solution to the problem.

Note that there is a simpler way to solve the problem, just using the fact that the solution is symmetric: we must have $d = \delta h$ and $l = \delta d$, from which the result follows. This does not make clear, however, where subgame perfection comes in.

7.11 The Little Miss Muffet Game

7.12 Nuisance Suits

7.13 Cooperation in an Overlapping-Generations Economy

Part (d): Let T^* be the age of veneration. Assume all members but one cooperate, and test whether the final player will cooperate. If this final player is of age $t \leq T^*$, the gains from cooperating are $(T - t + 1)nT^* - (T^* - t + 1)\alpha$, and the gains from defecting are $nT^* - 1$. Thus, the net gains from cooperating are

$$f(t, T^*) = (T - t)nT^* - (T^* - t + 1)\alpha + 1.$$

Then T^* supports a Nash subgame perfect equilibrium of the desired form if and only if $f(t, T^*) \geq 0$ for all $t = 1, \ldots, T^*$. In particular, we must have $f(T^*, T^*) \geq 0$. But $f(t, t) = (T - t)nt - (\alpha - 1)$ is a parabola with a maximum at $t = T/2$, and $f(T/2, T/2) = nT^2/4 - (\alpha - 1)$. Since this must be nonnegative, we see that

$$T^2 \geq \frac{4(\alpha - 1)}{n} \tag{A7.1}$$

is necessary.

Now suppose (7.1) holds, and choose T^* such that $f(T^*, T^*) \geq 0$. Since $f_1(t, T^*) = -nT^* + \alpha$, if $\alpha \leq nT^*$ then f is decreasing in t, so $f(t, T^*) \geq 0$ for all $t = 1, \ldots, T^*$. If $\alpha > nT^*$, $f(t, T^*)$ is increasing in t, so we must ensure that $f(1, T^*) \geq 0$. But $f(1, T^*) = T^*(n(T - 1) - \alpha) + 1$, which is strictly positive by assumption. Thus, (7.1) is sufficient.

Part (e): The total utility from the public good for a member is $T^*(nT - \alpha)$, which is an increasing function of T^*.

7.14 A Behavioral Approach to Backward Induction

7.15 The Centipede Game

7.16 The Finitely Repeated Prisoner's Dilemma

7.17 The Fallacy of Backward Induction

7.18 The Surprise Examination

7.19 An Agent-based Simulation of the 100-round Prisoner's Dilemma

7.20 Interactive Epistemology and Backward Induction

7.21 The Finitely Repeated Prisoner's Dilemma II

a. A string of k defections followed by a return to cooperation costs the player $k+1+(T-S)$, which is strictly positive for $k \geq 1$. To see this, we can clearly assume we are at round 1, since the argument is the same further down the line. Cooperating until the last round, then defecting, pays $n-1+T$, while defecting for k rounds and then cooperating returns $T+S+(n-k-2)+T$. You get T on the first round; you get nothing for $k-1$ rounds; you get S when you return to cooperate, then you get 1 for $n-k-2$ rounds; you defect on the last round and get T.

b. By preempting, player 1 is revealed to be a best responder (Reciprocators never preempt). The statement is clearly true if player 2 is a Reciprocator. If player 2 is a best responder, and if he does not defect, player 1 would know he is best responder. Since both players would now know they are not Reciprocators, they would both defect forever. Meanwhile, player 2 would get S on this round by cooperating. So player 2 should defect in response to preemption. Moreover, if later player 2 cooperated while player 1 continued to defect, player 1 would again know that player 2 is a best responder, and so both would defect forever. The assertion follows.

c. The best-case scenario is that when player 1 returns to C, player 2 does so as well. But in (a), we saw that player 1 can't gain from such a move.

d. We know that player 1 cannot gain by preempting at stage $k-1$ and returning to C at some later stage. Thus, by preempting at $k-1$, player 1 will defect thereafter in an optimal strategy. The payoff to this is T. But by waiting until next period to defect, he gets 1 now and ϵT next period. But by assumption $(1-\epsilon)T < 1$, so $1+\epsilon T > T$, so waiting dominates preempting.

e. We now know that a best responder should either cooperate until the last round, or preempt exactly when he expects his partner will preempt if his partner is a best responder. The total payoff to the latter strategy is $k-1+\epsilon T$ if he thinks a best responder will switch on round k, and the payoff to the former strategy is $(k-1+S)(1-\epsilon)+(n-1+T)\epsilon$. The preempt strategy is thus superior precisely when $k > n+S(1-\epsilon)/\epsilon$ and then only if the player believes a best responder will preempt on round k.

f. This follows from the fact that a best responder will never preempt.

7.22 Fuzzy Subgame Perfection

7.23 Perfect Behavioral Nash Equilibria

7.24 Selten's Horse

If Walter's information set is not reached with positive probability, an equilibrium must have the form $(A, a, p_\lambda\lambda + (1-p_\lambda)\rho)$. Franz's payoff to D is $4p_\lambda + (1-p_\lambda)$, which must be at most 3, so $p_\lambda \leq 2/3$. Gustav's payoff to d is $5p_\lambda + 2(1-p_\lambda)$, which must be at most 3, so $p_\lambda \leq 1/3$. This gives us the set M of equilibria.

Now assume Walter's information set is reached. If Walter chooses the pure strategy λ, either Gustav must not get to choose, or Gustav must choose d. If Gustav does not get to choose, Franz must choose D, with payoffs (4,4,4). If Gustav plays a with probability p_a, the payoff to A is $3p_a + 5(1-p_a)$, which must be at most 4, so $p_a \geq 1/2$. This gives us the set N of equilibria. If Gustav does get to choose, Gustav chooses d, so Franz must randomize or else Walter would not choose λ. But the payoff to A for Walter is 5, and the payoff to D is 4, so Franz will not randomize. Thus, there are no Nash equilibria in this category.

If Walter's information set is reached and Walter chooses pure strategy ρ, either Gustav does not get to choose, or Gustav chooses a. If Gustav does not get to choose, Franz chooses D, which is clearly inferior to A, no matter what Gustav's strategy is. If Gustav does choose, Gustav chooses a, so $p_A = 1$, which contradicts the assumption that Walter's information set is reached with positive probability.

The remaining case is that Walter's information set is reached with positive probability, and Walter randomizes. The payoff to λ is $4p_{3l}$, where p_{3l} is the probability of being at $3l$, and the payoff to ρ is $p_{3l}+2(1-p_{3l})$, and since these must be equal, we have $p_{3l} = 2/5$. But clearly, $p_{3l} = (1-p_A)/(1-p_A+p_A(1-p_a))$. This implies $p_A(5-2p_a) = 3$. Clearly, p_A cannot then be zero or one, so Franz randomizes, which implies

$$4p_\lambda + (1-p_\lambda) = 3p_a + (1-p_a)(5p_\lambda + 2(1-p_\lambda)),$$

or $p_a(3p_\lambda - 1) = 1$. If Gustav randomizes, we must have $3 = 5p_\lambda + 2(1-p_\lambda)$, so $p_\lambda = 1/3$, which is impossible. Thus, $p_a = 1$ and $p_\lambda = 1/3$, which is again impossible. So this case is impossible.

7.25 Trembling Hand Perfection

a. A straightforward calculation yields

$$p_{3l} = \frac{\epsilon_1^*}{\epsilon_1^* + \epsilon_2^*}$$

where $\epsilon_1^* = \epsilon_1 + (1 - 2\epsilon_1)(1 - p_A)$ and $\epsilon_2^* = \epsilon_1 + (1 - 2\epsilon_1)(1 - p_A) + (\epsilon_1 + (1 - 2\epsilon_1)p_A)(\epsilon_2 + (1 - 2\epsilon_2)(1 - p_a))$.

b. If $p_A < 1$, then p_{3l} approaches

$$p_{3l}^{\infty} = \frac{(1 - p_A)}{(1 - p_A) + p_A(1 - p_a)}$$

for small ϵ_1, ϵ_2. Suppose $p_A < 1$. Then $p_\lambda = 1$ for small ϵ_1, ϵ_2 if $p_{3l}^{\infty} \geq 2/5$, which reduces to $3 \geq p_A(5 - 2p_a)$. In this situation Gustav chooses d, so $p_a = 0$. The payoff of A to Walter is 5 and the payoff to D is 4, so $p_A = 1$, which is a contradiction. We next try $p_\lambda = 0$ for small ϵ_1, ϵ_2, can be true only if $3 \leq p_A(5 - 2p_a)$. Now $p_a = 1$, so $p_A = 1$, a contradiction. The only alternative if $p_A < 1$ is that Walter randomizes for small ϵ_1, ϵ_2, which means that $3 = p_A(5 - 2p_a)$. It follows that Franz randomizes as well, and the condition that the payoffs to A and D are equal for Franz reduces to $p_a(3p_\lambda - 1) = 1$. This implies $p_\lambda > 1/3$, so the payoff to d is greater than $5(1/3) + 2(2/3) = 3$, so $p_a = 0$, which is a contradiction.

We conclude that for sufficiently small ϵ_1, ϵ_2, $p_A = 1$, which implies

$$p_{3l} = \frac{\epsilon_1}{\epsilon_1 + (1 - \epsilon_1)(\epsilon_2 + (1 - 2\epsilon_2)(1 - p_a))}.$$

It is clear that in this case Walter cannot choose $p_\lambda = 1$ since then $p_a = 0 \to p_\lambda = 0$, a contradiction. If $p_\lambda = 0$ then $p_a = 1$, so

$$p_{3l} = \frac{\epsilon_1}{\epsilon_1 + (1 - \epsilon_1)\epsilon_2}.$$

This must be a local best response for Walter, which is the case so long as $p_{3l} \geq 2/5$, which is equivalent to $3\epsilon_1 \geq 2(1 - \epsilon_1)e_2$. If this inequality fails, then Walter must randomize, which means $p_{3l} = 2/5$, and given ϵ_1 and ϵ_2, this uniquely determines $p_a < 1$. In this case, as $\epsilon_1, \epsilon_2 \to 0$ while maintaining $3\epsilon_1 < (1 - \epsilon_1)e_2$, $p_a \to 1$, so in all cases for small ϵ_1, ϵ_2, the equilibrium is near the perfect equilibrium.

7.26 Nature Abhors Low Probability Events

8

Repeated Games, Trigger Strategies, and Tacit Collusion

8.1 Introduction

8.2 Big Fish and Little Fish

8.3 Tacit Collusion

8.4 The Folk Theorem: An Embarras de Richesses

8.5 Variations on the Folk Theorem

8.6 The One-Stage Deviation Principle

8.7 The Folk Theorem: A Behavioral Critique

8.8 A Trembling Hand, Cooperative Equilibrium

8.9 Death and Discount Rates in Repeated Games

You will get the answer approximately right if you set the problem up in any reasonable way. For an exact answer, write the present value v as

$$v = \frac{\pi + (1-\sigma)v}{1+\rho}.$$

8.10 The Strategy of an Oil Cartel

Payoffs:	Low/Low	$(25-2)\times 2, (25-4)\times 2$	=	46,42
	High/Low	$(15-2)\times 4, (15-4)\times 2$	=	52,22
	Low/High	$(15-2)\times 2, (15-4)\times 4$	=	26,44
	High/High	$(10-2)\times 4, (10-4)\times 4$	=	32,24

Normal Form Game:

	Low	High
Low	46,42	26,44
High	52,22	32,24

The condition for cooperate to beat defect for Iran is

$$\frac{46}{1-\delta} > 52 + \delta\frac{32}{1-\delta}.$$

We can solve this, getting $\delta > 0.3$, which corresponds to an interest rate r given by $r = (1-\delta)/\delta = 0.7/0.3 \approx 233.33\%$. The condition for cooperate to beat defect for Iraq is

$$\frac{42}{1-\delta} > 44 + \delta\frac{24}{1-\delta}.$$

We can solve this, getting $\delta > 0.1$, which corresponds to an interest rate r given by $r = 900\%$.

8.11 Manny and Moe

Let v be the present value of cooperating forever for Manny. Then $v = 3 + pv$, since cooperation pays 3, plus with probability p Manny gets the present value v again in the next period. Solving, we get $v = 3/(1-p)$. If Manny defects, he gets 5 now, and then 1 forever, starting in the next period. The value of getting 1 forever is $v_1 = 1 + p \cdot v_1$, so $v_1 = p/(1-p)$. Thus Manny's total return to defecting is $5+p/(1-p)$. Cooperating beats defecting for Manny if $3/(1-p) > 5+p/(1-p)$. Solving, we find Manny should cooperate as long as $p > 50\%$.

8.12 Tit-for-Tat

For parts (a) and (b), first show that tit-for-tat is not Nash when played against itself if and only if player 1 gains by defecting on the first move. Second, show that defecting once and going back to tit-for-tat increases payoff by $t + \delta s - r(1+\delta)$, which is negative for $\delta_1 = (t-r)/(r-s)$. Third, show that $0 < \delta_1 < 1$. Fourth, show that after the first defect, the gain from defecting for k more rounds before returning to cooperate is $\delta^{k+1}(p - s(1-\delta) - \delta r)$. If this is negative, then δ_1 is the minimum discount factor for which tit-for-tat is Nash against itself. If this is positive, show that defecting forever increases payoff by $t - r - (r-p)\delta/(1-\delta)$, which is negative for a discount factor greater that $\delta_2 = (t-r)/(t-p)$. Finally, show that $0 < \delta_2 < 1$.

8.13 Reputational Equilibrium

If it is worthwhile for the firm to lie when it claims its product has quality $q > 0$, it might as well set its actual quality to 0, since the firm minimizes costs this way. Its profits are then

$$\pi_f = (4 + 6q_a - x - 2)x = (2 + 6q_a - x)x.$$

Profits are maximized when

$$\frac{d\pi_f}{dx} = 2 + 6q_a - 2x = 0,$$

so $x = 1 + 3q_a$, and $\pi_f = (1+3q_a)^2$.

Now suppose the firm tells the truth. Then, if π_t is per-period profits, we have

$$\pi_t = (2 + 6q_a - 6q_a^2 - x)x,$$
$$\frac{d\pi_t}{dx} = 2 + 6q_a - 6q_a^2 - 2x = 0,$$

so $x = 1 + 3q_a - 3q_a^2$, and $\pi_t = (1 + 3q_a - 3q_a^2)^2$. But total profits Π from truth-telling are π_t forever, discounted at rate $\delta = 0.9$, or

$$\Pi = \frac{\pi_t}{1-\delta} = 10(1 + 3q_a - 3q_a^2)^2.$$

Truth-telling is profitable then when $\Pi \geq \pi_f$, or when

$$10(1 + 3q_a - 3q_a^2)^2 > (1 + 3q_a)^2. \tag{A8.1}$$

Note that equation (A8.1) is true for very small q_a (i.e., q_a near 0) and false for very large q_a (i.e., q_a near 1).

8.14 Contingent Renewal Contracts

8.15 Contingent Renewal Labor Markets

8.16 I'd Rather Switch than Fight

9

Biology Meets Economics: Evolutionarily Stable Strategies and the Birth of Dynamic Game Theory

9.1 The Birth of Evolutionarily Stable Strategies

9.2 Properties of Evolutionarily Stable Strategies

a. Simple

b. The two conditions for an evolutionarily stable strategy imply that if σ is an evolutionarily stable strategy, then $\pi_{\tau\sigma} \leq \pi_{\sigma\sigma}$ for any other strategy τ. But this is just the Nash condition.

c. If σ is a strict Nash equilibrium, then $\pi_{\tau\sigma} < \pi_{\sigma\sigma}$ for any $\tau \neq \sigma$, so σ is an evolutionarily stable strategy.

d. Suppose $\pi_{11} > \pi_{21}$. Then pure strategy 1 is a strict Nash equilibrium, so it is an evolutionarily stable strategy. The same is true if $\pi_{22} > \pi_{12}$. So suppose $\pi_{11} < \pi_{21}$ and $\pi_{22} < \pi_{12}$. Then we can show that the game has a unique completely mixed symmetric equilibrium p, where each player uses strategy 1 with probability $\alpha_p \in (0, 1)$. The payoff to strategy 1 against the mixed strategy $(\alpha_p, 1-\alpha_p)$ is then $\alpha_p\pi_{11} + (1-\alpha_p)\pi_{12}$, and the payoff to strategy 2 against this mixed strategy is $\alpha_p\pi_{21} + (1-\alpha_p)\pi_{22}$. Since these must be equal, we find that $\alpha_p = (\pi_{22} - \pi_{12})/\Delta$, where $\Delta = \pi_{11} - \pi_{21} + \pi_{22} - \pi_{12} < 0$. Note that under our assumptions, $0 < \alpha_p < 1$, so there is a unique completely mixed Nash equilibrium $(\alpha_p, 1 - \alpha_p)$. Is this an ESS?

   Let α_q be the probability a mutant player uses pure strategy 1. Since each pure strategy is a best response to α_p, α_q must also be a best response to α_p, so clearly, $\pi_{qp} = \pi_{pp}$. To show that p is an ESS, we must show that $\pi_{pq} > \pi_{qq}$. We have

$$\pi_{pq} = \alpha_p[a_{11}\alpha_q + \pi_{12}(1-\alpha_q)] + (1-\alpha_p)[\pi_{21}\alpha_q + \pi_{22}(1-\alpha_q)]$$

and

$$\pi_{qq} = \alpha_q[\pi_{11}\alpha_q + \pi_{12}(1-\alpha_q)] + (1-\alpha_q)[\pi_{21}\alpha_q + \pi_{22}(1-\alpha_q)].$$

Subtracting and simplifying, we get

$$\pi_{pq} - \pi_{qq} = -(\alpha_p - \alpha_q)^2\Delta > 0,$$

which proves we have an ESS.

e. It is easy to check that if there is a mixed strategy equilibrium, the frequency α of pure strategy 1 must satisfy

$$\alpha = \frac{\pi_{22} - \pi_{12}}{\Delta}, \qquad \text{where } \Delta = \pi_{11} - \pi_{21} + \pi_{22} - \pi_{12}.$$

Suppose $\Delta > 0$. Then $0 < \alpha < 1$ if and only if $0 < \pi_{22} - \pi_{12} < \pi_{11} - \pi_{21} + \pi_{22} - \pi_{12}$, which is true if and only if $\pi_{11} > \pi_{21}$ and $\pi_{22} > \pi_{12}$. If $\Delta < 0$, a similar argument shows that $0 < \alpha < 1$ if and only if the other pair of inequalities holds.

Suppose there is a "mutant" that uses pure strategy 1 with probability β. Thus, in general,

$$\begin{aligned}\pi_{\gamma\delta} &= \gamma\delta\pi_{11} + \gamma(1-\delta)\pi_{12} + (1-\gamma)\delta\pi_{21} + (1-\gamma)(1-\delta)\pi_{22} \\ &= \gamma\delta\Delta + \delta(\pi_{21} - \pi_{22}) + \gamma(\pi_{12} - \pi_{22}) + \pi_{22}.\end{aligned}$$

It follows that

$$\pi_{\alpha\alpha} - \pi_{\beta\alpha} = (\alpha - \beta)[\alpha\Delta - (\pi_{22} - a_{12})] = 0,$$

so the equilibrium is an ESS if and only if $\pi_{\alpha\beta} > \pi_{\beta\beta}$. But

$$\begin{aligned}\pi_{\alpha\beta} - \pi_{\beta\beta} &= \alpha\beta\Delta + \beta(a_{21} - a_{22}) + \alpha(a_{12} - a_{22}) + a_{22} \\ &\qquad - \beta^2\Delta - \beta(a_{21} - a_{22}) - \beta(a_{12} - a_{22}) - a_{22} \\ &= \beta(\alpha - \beta)\Delta + (\alpha - \beta)(a_{12} - a_{22}) \\ &= (\alpha - \beta)(\beta\Delta + a_{12} - a_{22}) \\ &= (\alpha - \beta)(\beta\Delta - \alpha\Delta) \\ &= -(\alpha - \beta)^2\Delta.\end{aligned}$$

Thus, the equilibrium is an ESS if and only if $\Delta < 0$, which is equivalent to $a_{11} < a_{21}$ and $a_{22} < a_{12}$. This proves the assertion.

f. Suppose there are an infinite number of distinct evolutionarily stable strategies. Then there must be two, say σ and τ, that use exactly the same pure strategies. Now τ is a best response to σ, so σ must do better against τ than τ does against itself. But σ does equally well against τ as τ does against τ. Thus, σ is not an ESS and similarly for τ.

g. First, suppose σ is an ESS, so for any $\tau \neq \sigma$, there is an $\tilde{\epsilon}(\tau)$ such that[1]

$$\pi_{\tau,(1-\epsilon)\sigma+\epsilon\tau} < \pi_{\sigma,(1-\epsilon)\sigma+\epsilon\tau} \quad \text{for all } \epsilon \in (0, \tilde{\epsilon}(\tau)). \tag{A9.1}$$

In fact, we can choose $\tilde{\epsilon}(\tau)$ as follows. If (A9.1) holds for all $\epsilon \in (0, 1)$, then let $\tilde{\epsilon}(\tau) = 1$. Otherwise, let $\tilde{\epsilon}$ be the smallest $\epsilon > 0$ such that (A9.1) is violated and define

$$\tilde{\epsilon}(\tau) = \frac{\pi_{\sigma\sigma} - \pi_{\tau\sigma}}{\pi_{\tau\tau} - \pi_{\tau\sigma} - \pi_{\sigma,\tau} + \pi_{\sigma\sigma}}.$$

It is easy to check that $\tilde{\epsilon}(\tau) \in (0, 1]$ and (A9.1) are satisfied. Let $T \subset S$ be the set of strategies such that if $\tau \in T$, then there is at least one pure strategy used in σ that is not used in τ. Clearly, T is closed and bounded, $\sigma \notin T$, $\tilde{\epsilon}(\tau)$ is continuous, and $\tilde{\epsilon}(\tau) > 0$ for all $\tau \in T$. Hence, $\tilde{\epsilon}(\tau)$ has a strictly positive minimum ϵ^* such that (A9.1) holds for all $\tau \in T$ and all $\epsilon \in (0, \epsilon^*)$.

If τ is a mixed strategy and s is a pure strategy, we define $s(\tau)$ to be the weight of s in τ (i.e., the probability that s will be played using τ). Now consider the neighborhood of s consisting of all strategies τ such that $|1-s(\tau)| < \epsilon^*$ for all pure strategies s. If $\tau \neq s$, then $\epsilon^* > 1-s(\tau) = \epsilon > 0$ for some pure strategy s. Then $\tau = (1-\epsilon)s + \epsilon r$, where $r \in T$. But then (A9.1) gives $\pi_{r\tau} < \pi_{s,\tau}$. If we multiply both sides of this inequality by ϵ and add $(1-\epsilon)\pi_{s\tau}$ to both sides, we get $\pi_{\tau\tau} < \pi_{s,\tau}$, as required. The other direction is similar, which proves the assertion.

h. If σ is completely mixed, then for any $t \in S$, $\pi_{\sigma\sigma} = \pi_{t\sigma}$, simply because any pure strategy has the same payoff against σ as σ does against σ. Therefore, any mixed strategy has the same payoff against σ as σ has against σ. For similar reasons, $\pi_{\sigma\tau} = \pi_{\sigma\sigma}$. Thus, σ is an ESS and if τ is any other strategy, we must have $\pi_{\sigma\tau} > \pi_{\tau\tau}$.

[1] We write (a, b) to mean the set of numbers $\{x|a < x < b\}$ and we call this the *open interval* (a, b).

9.3 Evolutionarily Stable Strategies: Basic Examples

a. Let s_a and s_b be the two strategies, and write $\sigma = \alpha s_a + (1-\alpha)s_b$ for the mixed strategy where s_a is played with probability α. If $\tau = \beta s_a + (1-\beta)s_b$, we have $\pi[\sigma, \tau] = \alpha\beta a + (1-\alpha)(1-\beta)b$. Suppose (σ, σ) is a Nash equilibrium. Then by the Fundamental Theorem (§6.4), $\pi[s_a, \sigma] = \pi[s_b, \sigma]$, which implies $\alpha = b/(a+b)$. Note that $\pi[\sigma, \sigma] = ab/(a+b)$, which is smaller than either a or b. We shall show that b can invade a population that plays σ. By the Fundamental Theorem, $\pi[s_b, \sigma] = \pi[\sigma, \sigma]$, since $\alpha < 1$. Thus σ is impervious to invasion by s_b only if $\pi[\sigma, s_b] > \pi[s_b, s_b]$, which reduces to $ab/(a+b) > b$, which is false.

9.4 Hawks, Doves, and Bourgeois

a. The payoff to H is $\alpha(v-w)/2 + (1-\alpha)v = v - \alpha(v+w)/2$, and the payoff to D is $(1-\alpha)(v/2) = v/2 - \alpha(v/2)$. These are equated when $\alpha = v/w$, which is < 1 if $w > v$. To show that this mixed strategy equilibrium is an ESS, we can refer to §9.2e. In this case $\pi_{11} = (v-w)/2$, $\pi_{21} = 0$, $\pi_{22} = v/2$, and $\pi_{12} = v$. Thus $\pi_{11} = (v-w)/2 < 0 = \pi_{21}$ and $\pi_{22} = v/2 < v = \pi_{12}$, so the equilibrium is an ESS.

b. Since the payoff to Bourgeois against Bourgeois, $v/2 > 3v/4 - w/4$, which is the payoff to Hawk against Bourgeois, and $v/2 > v/4$, which is the payoff to Dove against Bourgeois, Bourgeois is a strict Nash equilibrium, and hence is an ESS.

9.5 Trust in Networks II

For specificity, we take $p = 0.8$. You can check that the equilibrium has Inspect share $\alpha^* \approx 0.71$ Trust share $\beta^* \approx 0.19$, and Defect share $\gamma^* \approx 0.10$. The payoff to the equilibrium strategy s is $\pi_{ss} \approx 0.57$. The payoff to Trust against the equilibrium strategy is of course $\pi_{ts} = \pi_{ss} \approx 0.57$, but the payoff to Trust against itself is $\pi_{tt} = 1$, so Trust can invade.

9.6 Cooperative Fishing

Here is the normal form game.

	Put Out	Free Ride
Put Out	$\frac{\nu}{2}-c_2, \frac{\nu}{2}-c_2$	$\frac{\nu}{2}-c_1, \frac{\nu}{2}$
Free Ride	$\frac{\nu}{2}, \frac{\nu}{2}-c_1$	0,0

It is easy to see there are no pure strategy symmetric equilibria, since $\nu/2 > c_1$. There are two pure strategy asymmetric equilibria, FP and PF. Consider a mixed strategy equilibrium where a fraction α of the population plays P. The payoff to P is then

$$\alpha\left(\frac{\nu}{2}-c_2\right)+(1-\alpha)\left(\frac{\nu}{2}-c_1\right)=\frac{\nu}{2}-[\alpha c_2+(1-\alpha)c_1].$$

The payoff to F is simply $\alpha\nu/2$. Equating the two payoffs, we get

$$\alpha=\frac{\frac{\nu}{2}-c_1}{\frac{\nu}{2}+c_2-c_1}.$$

Note that we have $0<\alpha<1$, so this is a strictly mixed Nash equilibrium. Is this mixed strategy, which we will call M, an evolutionarily stable strategy? We can again refer to §9.2e. In this case $\pi_{11}=v/2-c_2$, $\pi_{21}=v/2$, $\pi_{22}=0$, and $\pi_{12}=v/2-c_1$. Thus $\pi_{11}=v/2-c_2<v/2=\pi_{21}$ and $\pi_{22}=0<v/2-c_1=\pi_{12}$, so the equilibrium is an ESS.

9.7 Are Evolutionarily Stable Strategies Unbeatable?

9.8 A Nash Equilibrium That Is Not Evolutionarily Stable

9.9 Rock, Paper, and Scissors Is Not an ESS

9.10 Sex Ratios as Evolutionarily Stable Strategies

9.11 Invasion of the Pure Strategy Mutants

9.12 Multiple Evolutionarily Stable Strategies

9.13 The Logic of Animal Conflict

9.14 Trogs and Farfel

a. The payoff matrix to the game is as follows.

	Good	Bad
Good	$z-y, z-y$	$w-y, z-x$
Bad	$z-x, w-y$	$w-x, w-x$

By definition, a 2×2 prisoner's dilemma is given by

	C	D
C	r, r	s, t
D	t, s	p, p

where $t > r > p > s$ and $r > (s+t)/2$. This is true if and only if $z - w > y - x$, as can be seen from the following. In our case, $t > r > p > s$ becomes $z - x > z - y > w - x > w - y$. The first and third inequalities are always true, and the second is equivalent to $z - w > y - x$. Also, $r > (s+t)/2$ is equivalent to $2z - 2y > z - x + w - y$, which is equivalent to $z - y > w - x$.

b. Obvious, since "always give good farfel" leads to cooperation on every round, either against itself or against Tit-for-Tat.

c. First, we show that playing B forever is better than Tit-for-Tat if and only if $\delta < (y-x)/(z-w)$. Tit-for-Tat against Tit-for-Tat gives $z - y$ for all rounds, with present value $(z-y)/(1-\delta)$. The present value of defecting forever is $(z-x) + \delta(w-x)/(1-\delta)$. But $(z-y)/(1-\delta) > (z-x) + \delta(w-x)/(1-\delta)$ if and only if $z - y > (1-\delta)(z-x) + \delta(w-x) = z - x - \delta z + \delta w$ if and only if $y - x < \delta(z-w)$ if and only if $\delta > (y-x)/(z-w)$.

Next, we show that playing B for a certain number of periods, say $n \geq 1$, followed by a G is not a best response to Tit-for-Tat if and only if the above inequality holds. The history of play then looks like

$$GGG...BB.....BGxxxxx...$$
$$GGG...GB.....BBGxxxx...,$$

where we don't know what the x's are. Let us compare this to what happens if player 1 defects one fewer time, so we now have

$$GGG...BB.....GGxxxxx....$$
$$GGG...GB.....BGGxxxx....$$

The only changes in the payoff to player 1 are in periods $n-1$ and n. Player 1 gains $(w-y)-(w-x)=x-y$ (which is negative) in period $n-1$ and gains $(z-y)-(w-y)=z-w$ in period n. The total gain, discounted to period $n-1$, is then $x-y+\delta(z-w)$, which is positive if and only if the above inequality holds.

9.15 Evolutionarily Stable Strategies in Finite Populations

a. Let $r_{\mu\nu}=0$, $r_{\nu\mu}=n$, $r_{\mu\mu}=n+1$, and $r_{\nu\nu}=-1$. Then ν is not Nash, since $r_{\mu\nu}>r_{\nu\nu}$, μ is Nash since $r_{\mu\mu}>r_{\mu\nu}$ and $r_{\mu\mu}>r_{\nu\mu}$, but $r(\nu)-r(\mu)=1/n>0$ for any m.
b. This is trivial if there is an alternate best response to a Nash strategy.
c. Suppose ν is evolutionarily stable but is not Nash. Then there is some μ such that $r_{\nu\nu}<r_{\mu\nu}$. Let $m=1$. Then for sufficiently large n we have $r(\nu)<r(\mu)$ in

$$r(\nu)-r(\mu)=\left(1-\frac{m}{n}\right)(r_{\nu\nu}-r_{\mu\nu})+\frac{m}{n}(r_{\nu\mu}-r_{\mu\mu})+\frac{1}{n}(r_{\mu\mu}-r_{\mu\nu}). \tag{A9.2}$$

Hence, ν must be Nash. Now suppose ν is evolutionarily stable and $r_{\nu\nu}=r_{\mu\nu}$ but $r_{\nu\mu}<r_{\mu\mu}$. Equation (A9.2) becomes

$$r(\nu)-r(\mu)=\frac{1}{n}\left\{m[r_{\nu\mu}-r_{\mu\mu}]+[r_{\mu\mu}-r_{\mu\nu}]\right\}.$$

Given $\epsilon>0$, choose $\overline{m}$ so that the term in brackets is negative, and then choose n so that $\overline{m}/n<\epsilon$. Then $r(\nu)<r(\mu)$ for all positive $m\leq\overline{m}$, which is a contradiction. So suppose in addition to $r_{\nu\nu}=r_{\mu\nu}$ and $r_{\nu\mu}=r_{\mu\mu}$, we have $r_{\mu\mu}<r_{\mu\nu}$. Then clearly $r(\nu)-r(\mu)=[r_{\mu\mu}-r_{\mu\nu}]/n<0$, again a contradiction. This proves that the stated conditions are necessary. We can reverse the argument to prove the conditions are sufficient as well.
d. In the limit we have

$$r(\nu)-r(\mu)=(1-\epsilon)[r_{\nu\nu}-r_{\mu\nu}]+\epsilon[r_{\nu\mu}-r_{\mu\mu}].$$

The conclusion follows immediately from this equation. The limit argument cannot be used to conclude that $r(\nu)>r(\mu)$ in the "large finite" case if $r_{\nu\nu}=r_{\mu\nu}$ and $r_{\nu\mu}=r_{\mu\mu}$.

9.16 Evolutionarily Stable Strategies in Asymmetric Games

10

Dynamical Systems and Differential Equations

10.1 Introduction

10.2 Dynamical Systems

10.3 Population Growth

10.4 Population Growth with Limited Carrying Capacity

10.5 The Lotka-Volterra Predator-Prey Model

10.6 Dynamical Systems Theory

10.7 Dynamical Systems in One Dimension

10.8 Dynamical Systems in Two Dimensions

10.9 Exercises in Two-Dimensional Linear Systems

10.10 Cultural Dynamics

After the substitution $z = 1 - x - y$, the Jacobian is $\begin{pmatrix} -\alpha - \gamma & -\gamma \\ \alpha & -\beta \end{pmatrix}$. The eigenvalues of this matrix are

$$\frac{1}{2}\left(-\delta \pm \sqrt{\delta^2 - 4(\alpha\beta + \alpha\gamma + \beta\gamma)}\right),$$

where $\delta = \alpha + \beta + \gamma$. Suppose $\gamma < \beta$ (the other cases are similar). Then the equilibrium is a stable focus for $(1 - \gamma/\beta)^2 < \alpha/\beta < (1 + \gamma/\beta)^2$ and a stable node when $\alpha/\beta < (1 - \gamma/\beta)^2$ or $\alpha/\beta > (1 + \gamma/\beta)^2$. We cannot determine the behavior of the system at the two remaining points $\alpha = \beta(1 \pm \gamma/\beta)^2$, since the system is not hyperbolic there.

10.11 Lotka-Volterra with Limited Carrying Capacity

The interior evolutionary equilibrium satisfies

$$x^* = \frac{c}{d},$$

$$y^* = \frac{ad - c\epsilon}{bd}.$$

The Jacobian, evaluated at (x^*, y^*) is

$$J = \begin{bmatrix} -c\epsilon/d & -bc/d \\ y^* & 0 \end{bmatrix}$$

The eigenvalues of the Jacobian at the equilibrium are

$$\frac{-c\epsilon \pm \sqrt{c}\sqrt{c\epsilon^2 + 4cd\epsilon - 4ad^2}}{2d}.$$

When ϵ is small, the term under the square root sign is negative, so both eigenvalues have negative real parts. The equilibrium in this case is a stable node. If ϵ is large, but $\epsilon > ad/c$, it is easy to show that both eigenvalues are negative, so the equilibrium is a stable node.

10.12 Take No Prisoners

10.13 The Hartman-Grobman Theorem

10.14 Special Features of Two-Dimensional Dynamical Systems

10.15 A Non-Hyperbolic Dynamical System

10.16 Liapunov's Theorem

11

Evolutionary Dynamics

11.1 Introduction

11.2 The Origins of Evolutionary Dynamics

11.3 Properties of the Replicator System

Only the last part of the question might not be obvious. Let $p(t)$ be a trajectory of (11.5), and define

$$b(t) = \int_0^t \frac{dt}{a(p(t),t)},$$

which is possible since $a(p,t) > 0$. Clearly, $b(t)$ is positive and increasing. Let $q(t) = p(b(t))$. Then, by the Fundamental Theorem of the Calculus,

$$\begin{aligned}\dot{q}_i(t) &= \dot{b}(t)\dot{p}_i(b(t)) = \frac{1}{a(t)}a(t)p_i(b(t))(\pi_i(p(b(t))) - \bar{\pi}(p(b(t))))\\ &= q_i(t)(\pi_i(q(t)) - \bar{\pi}(q(t))). \blacksquare\end{aligned}$$

11.4 Characterizing the Two-Variable Replicator Dynamic

11.5 Do Dominated Strategies Survive under a Replicator Dynamic?

11.6 Equilibrium and Stability with a Replicator Dynamic

11.7 Evolutionary Stability and Evolutionary Equilibrium

11.8 Characterizing Two-player Symmetric Games II

11.9 Trust in Networks III

You can check that the eigenvalues of the Jacobian at the equilibrium are given by

$$\lambda_1, \lambda_2 = \frac{-2 + 5p - 4p^2 + p^3}{2(1+p)(3p-1)} \pm \frac{\sqrt{4 - 60p + 177p^2 - 116p^4 - 110p^4 + 104p^5 + p^6}}{2(1+p)(3p-1)}.$$

This is pretty complicated, but you can check that the expression under the radical is negative for p near unity: factor out $(p-1)$ and show that the other factor has value 32 when $p = 1$. The rest of the expression is real and negative for p near unity, so the equilibrium is a stable focus.

11.10 Bayesian Perfection and Stable Sets

Use the following definitions,

$$\begin{aligned}
\pi_D &= 4p_\lambda + (1 - p_\lambda)\\
\pi_A &= (1 - p_a)(5p_\lambda + 2(1 - p_\lambda)) + 3p_a\\
\pi_d &= (1 - p_A)(4p_\lambda + (1 - p_\lambda)) + p_A(5p_\lambda + 2(1 - p_\lambda))\\
\pi_a &= (1 - p_A)(4p_\lambda + (1 - p_\lambda)) + 3p_A\\
\pi_\lambda &= 4(1 - p_A)\\
\pi_\rho &= (1 - p_A) + 2p_A(1 - p_a)\\
\pi_1 &= p_A\pi_A + (1 - p_A)\pi_D\\
\pi_2 &= p_a\pi_a + (1 - p_a)\pi_d\\
\pi_3 &= p_\lambda\pi_\lambda + (1 - p_\lambda)\pi_\rho\\
\dot{p}_A &= p_A(\pi_A - \pi_1)\\
\dot{p}_a &= p_a(\pi_a - \pi_2)\\
\dot{p}_\lambda &= p_\lambda(\pi_\lambda - \pi_3)
\end{aligned}$$

and a bit of algebra.

11.11 Invasion of the Pure Strategy Mutants, II

11.12 A Generalization of Rock, Paper, and Scissors

Note first that no pure strategy is Nash. If one player randomizes between two pure strategies, the other can avoid the -1 payoff, so only strictly mixed solutions can be Nash. Check that the only such strategy σ that is Nash uses probabilities (1/3,1/3,1/3). This is not evolutionarily stable for $\alpha < 0$, however, since the pure strategy R has payoff $\alpha/3$ against σ, which is also the payoff to σ against σ, and has payoff α against itself.

The payoff of the strategies against $(x_1, x_2, 1 - x_1 - x_2)$ are

$$\begin{aligned} R\colon & \quad \alpha x_1 + x_2 - (1 - x_1 - x_2) &=& \quad (1+\alpha)x_1 + 2x_2 - 1 \\ P\colon & \quad -x_1 + \alpha x_2 + (1 - x_1 - x_2) &=& \quad -2x_1 - (1-\alpha)x_2 + 1 \\ S\colon & \quad x_1 - x_2 + \alpha(1 - x_1 - x_2) &=& \quad (1-\alpha)x_1 - (\alpha+1)x_2 + \alpha \end{aligned}$$

The average payoff is then $2\alpha(x_1^2 + x_1x_2 + x_2^2 - x_1 - x_2) + \alpha$, and the fitnesses of the three types are

$$\begin{aligned} f_1\colon & \quad (1+3\alpha)x_1 + 2(1+\alpha)x_2 - (1+\alpha) - 2\alpha(x_1^2 + x_1x_2 + x_2^2) \\ f_2\colon & \quad -2(1-\alpha)x_1 - (1-3\alpha)x_2 + (1-\alpha) - 2\alpha(x_1^2 + x_1x_2 + x_2^2) \\ f_3\colon & \quad (1+\alpha)x_1 - (1-\alpha)x_2 - 2\alpha(x_1^2 + x_1x_2 + x_2^2). \end{aligned}$$

Note that $x_1 = x_2 = 1/3$ gives $f_1 = f_2 = f_3 = 0$, so this is our Nash equilibrium. For the replicator dynamic, we have $\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0$, so we only need the first two equations. Assuming $x_1, x_2 > 0$, we get

$$\begin{aligned} \frac{\dot{x}_1}{x_1} &= -(2\alpha(x_1^2 + x_1x_2 + x_2^2) - (1+3\alpha)x_1 - 2(1+\alpha)x_2 + (1+\alpha)) \\ \frac{\dot{x}_2}{x_2} &= -(2\alpha(x_1^2 + x_1x_2 + x_2^2) + 2(1-\alpha)x_1 + (1-3\alpha)x_2 - (1-\alpha)). \end{aligned}$$

It is straightforward to check that $x_1 = x_2 = 1/3$ is the only fixed point for this set of equations in the positive quadrant.

The Jacobian of this system at the Nash equilibrium is

$$\frac{1}{3}\begin{bmatrix} 1+\alpha & 2 \\ -2 & -1+\alpha \end{bmatrix}.$$

This has determinant $\beta = 1/3 + \alpha^2/9 > 0$, the trace is $\mathrm{Tr} = 2\alpha/3$ and the discriminant is $\gamma = \mathrm{Tr}^2/4 - \beta = -1/3$. The eigenvalues are thus $\alpha/3 \pm \sqrt{-3}/3$, which have nonzero real parts for $\alpha \neq 0$. Therefore, the system is hyperbolic. By Theorem 10.5, the dynamical system is a stable focus for $\alpha < 0$ and an unstable focus for $\alpha > 0$.

11.13 *Uta stansburia* in Motion

It is easy to check that if the frequencies of orange-throats (Rock), blue-throats (Paper), and yellow-striped (Scissors) are α, β, and $1-\alpha-\beta$, respectively, the payoffs to the three strategies are $1-\alpha-2\beta$, $2\alpha+\beta-1$, and $\beta-\alpha$, respectively. The average payoff is zero (check this!), so the replicator dynamic equations are

$$\begin{aligned} \frac{d\alpha}{dt} &= \alpha(1-\alpha-2\beta) \\ \frac{d\beta}{dt} &= \beta(2\alpha+\beta-1). \end{aligned} \tag{A11.1}$$

The Jacobian matrix at the fixed point $\alpha=\beta=1/3$ is given by

$$\begin{bmatrix} -1/3 & -2/3 \\ 2/3 & 1/3 \end{bmatrix}.$$

The trace of the Jacobian is thus zero, the determinant is $1/3>0$, and the discriminant is $-1/3<0$. By Theorem 10.5 the eigenvalues are imaginary so the system is not hyperbolic. It is easy to solve for the trajectories of this system—by Theorem 10.5 they are closed orbits and the fixed point is a center. But this tells us nothing about the original, nonlinear system (11.1), since the fixed point is not hyperbolic (see Theorem 10.3). So, back to the drawing board.

Let $V(\alpha,\beta,\gamma)=\ln(\alpha)+\ln(\beta)+\ln(\gamma)$. Along a trajectory of the dynamical system we have

$$\begin{aligned} \dot V &= \frac{\dot\alpha}{\alpha}+\frac{\dot\beta}{\beta}+\frac{\dot\gamma}{\gamma} \\ &= (1-\alpha-2\beta)+(2\alpha+\beta-1)+(\beta-\alpha)=0. \end{aligned}$$

Thus, V is constant on trajectories. This implies that trajectories are bounded and bounded away from $(0,0)$ so the set Γ of ω-limit points of a trajectory contains no fixed points, and hence by the Poincaré-Bendixson Theorem (Theorem 10.8), Γ is a periodic orbit. But then by Theorem 10.9, Γ must contain $(0,0)$. Hence, trajectories also must spiral around $(0,0)$, and since V is increasing along a ray going northeast from the fixed point, trajectories must be closed orbits.

11.14 The Dynamics of Rock-Paper-Scissors and Related Games

Let π_α, π_β, and π_γ be the payoffs to the three strategies. Then, we have

$$\pi_\alpha=\beta r+(1-\alpha-\beta)s=\beta(r-s)-\alpha s+s,$$

$$\pi_\beta = \alpha s + (1-\alpha-\beta)r = \alpha(s-r) - \beta r + r,$$
$$\pi_\gamma = \alpha r + \beta s.$$

It is easy to check that the average payoff is then

$$\begin{aligned}\bar{\pi} &= \alpha\pi_\alpha + \beta\pi_\beta + (1-\alpha-\beta)\pi_\gamma \\ &= (r+s)(\alpha+\beta-\alpha^2-\alpha\beta-\beta^2).\end{aligned}$$

At any fixed point involving all three strategies with positive probability, we must have $\pi_\alpha = \pi_\beta = \pi_\gamma$. Solving these two equations, we find $\alpha = \beta = \gamma = 1/3$, which implies that $\bar{\pi} = (r+s)/3$.

In a replicator dynamic, we have

$$\begin{aligned}\dot{\alpha} &= \alpha(\pi_\alpha - \bar{\pi}), \\ \dot{\beta} &= \beta(\pi_\beta - \bar{\pi}).\end{aligned}$$

Expanding these equations, we get

$$\begin{aligned}\dot{\alpha} &= -2\alpha\beta s - (r+2s)\alpha^2 + \alpha s + \alpha p(\alpha,\beta), \\ \dot{\beta} &= -2\alpha\beta r - (2r+s)\beta^2 + \beta r + \beta p(\alpha,\beta),\end{aligned}$$

where $p(\alpha,\beta) = (r+s)(\alpha^2+\alpha\beta+\beta^2)$.

This is of course a nonlinear ordinary differential equation in two unknowns. It is easy to check that its unique fixed point for $\alpha, \beta > 0$ is $\alpha = \beta = 1/3$, the mixed strategy Nash equilibrium for this game.

For the dynamics, we linearize the pair of differential equations by evaluating the Jacobian matrix of the right-hand sides at the fixed point. The Jacobian is

$$J(\alpha,\beta) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned}a_{11} &= -2\beta s - 2\alpha(r+2s) + s + p(\alpha,\beta) + \alpha(2\alpha+\beta)(r+s), \\ a_{12} &= -2\alpha s + \alpha(\alpha+2\beta)(r+s), \\ a_{21} &= -2\beta r + \beta(2\alpha+\beta)(r+s), \\ a_{22} &= r - 2\alpha r - 2\beta(2r+s) + p(\alpha,\beta) + \beta(\alpha+2\beta)(r+s),\end{aligned}$$

so

$$J(1/3, 1/3) = \frac{1}{3}\begin{pmatrix} -s & r-s \\ s-r & -r \end{pmatrix}.$$

The eigenvalues of the linearized system are thus

$$\frac{1}{6}\left[-(r+s) \pm i\sqrt{3}(r-s)\right].$$

We prove the assertions as follows:

a. The determinant of the Jacobian is $(r^2 - rs + s^2)/9$. This has a minimum where $2s - r = 0$, with the value $r^2/12 > 0$. This shows that the system is hyperbolic, and since the determinant is positive, it is a node or a focus.
b. The real parts of the eigenvalues are negative if and only if $r + s > 0$, and are positive if and only if $r + s < 0$.
c. The eigenvalues are complex for $r \neq s$.
d. If $r + s = 0$, the eigenvalues are purely imaginary, so origin is a center. We thus cannot tell how the nonlinear system behaves using the linearization.

   However, we can show that the quantity $q(\alpha, \beta) = \alpha\beta(1 - \alpha - \beta)$ is constant along trajectories of the dynamical system. Assuming this (which we will prove in a moment), we argue as follows. Consider a ray R through the fixed point (1/3,1/3) pointing in the α-direction. Suppose $q(\alpha, \beta)$ is strictly decreasing along this ray (we will also prove this in a moment). Then, the trajectories of the dynamical system must be closed loops. To see this, note first that the fixed point cannot be a *stable node*, since if we start at a point on R near the fixed point, q decreases as we approach the fixed point, but q must be constant along trajectories, which is a contradiction. Thus, the trajectories of the system must be spirals or closed loops. But they cannot be spirals, because when they intersect R twice near the fixed point, the intersection points must be the same, since $q(\alpha, \beta)$ is constant on trajectories but decreasing on R near the fixed point.

   To see that q is decreasing along R near note that the differential equations for the dynamical system, assuming $r = 1$, $s = -1$, can be written as

$$\dot{\alpha} = 2\alpha\beta + \alpha^2 - \alpha \tag{A11.2}$$
$$\dot{\beta} = -2\alpha\beta - \beta^2 + \beta. \tag{A11.3}$$

   Then,

$$\begin{aligned}\frac{d}{dt}q(\alpha, \beta) &= \frac{d}{dt}[\alpha\beta(1 - \alpha - \beta)] \\ &= \beta(1 - \alpha - \beta)\dot{\alpha} + \alpha(1 - \alpha - \beta)\dot{\beta} + \alpha\beta(-\dot{\alpha} - \dot{\beta}) \\ &= 0,\end{aligned}$$

where we get the last step by substituting the expressions for $\dot{\alpha}$ and $\dot{\beta}$ from (A11.2) and (A11.3).

11.15 The Lotka-Volterra Model and Biodiversity

a. This is simple algebra, though you should check that the restrictions on the signs of a, b, c,and d ensure that $p^* > 0$.
b. We have

$$\begin{aligned}\frac{\dot{p}}{p} &= \frac{\dot{u}}{u} - \left[\frac{\dot{u}}{w} + \frac{\dot{v}}{w}\right] \\ &= \frac{\dot{u}}{u} - \left[p\frac{\dot{u}}{u} + (1-p)\frac{\dot{v}}{v}\right] \\ &= ap + b(1-p) - kw \\ &\quad - [p[ap + b(1-p) - kw] + (1-p)[cp + d(1-p) - kw]] \\ &= ap + b(1-p) - [p[ap + b(1-p)] + (1-p)[cp + d(1-p)]] \\ &= \pi_A - \bar{\pi}.\end{aligned}$$

c. We have

$$\begin{aligned}\frac{\dot{p}_i}{p_i} &= \frac{\dot{u}_i}{u_i} - \sum_{j=1}^{n} \frac{\dot{u}_j}{u} \\ &= \frac{\dot{u}_i}{u_i} - \sum_{j=1}^{n} \frac{\dot{u}_j}{u_j} p_j \\ &= \sum_{j=1}^{n} a_{ij} p_j - kw - \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{jk} p_k - ku\right) p_j \\ &= \sum_{j=1}^{n} a_{ij} p_j - ku - \sum_{j,k=1}^{n} a_{jk} p_k p_j + ku \sum_{k=1}^{n} p_j \\ &= \sum_{j=1}^{n} a_{ij} p_j - \sum_{j,k=1}^{n} a_{jk} p_j p_k.\end{aligned}$$

This proves the assertion, and the identification of the resulting equations as a replicator dynamic is clear from the derivation.

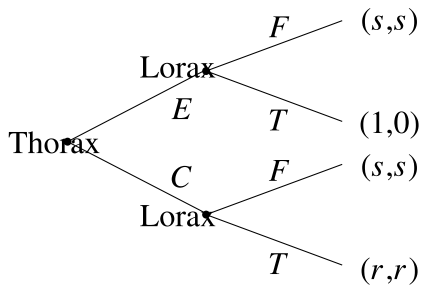
11.16 Asymmetric Evolutionary Games

11.17 Asymmetric Evolutionary Games: Reviewing the Troops

11.18 The Evolution of Trust and Honesty

11.19 The Loraxes and Thoraxes

a. We have the following game tree, on which backward induction (pruning the game tree) gives the desired result.



b. It is easy to show that there is no pure strategy Nash equilibrium other than FE, so we look for a mixed strategy Nash equilibrium. Let p be the fraction of Thoraxes who cooperate. Then, the payoff to T is $pr + (1-p)\cdot 0 = pr$, and the payoff to F is s, so in mixed strategy equilibrium we must have $pr = s$. Let q be the fraction of trusters among the Lorax. The payoff to C is then $qr + (1-q)s$, and the payoff to E is $q + (1-q)s$. Thus, Thoraxes always eat, so $p = 1$, which contradicts $pr = s$.

c. The payoff to I is $pr + (1-p)s - \delta$ for a Lorax, and the payoff to NI is $\max\{pr, s\}$, since if $pr > s$, the noninspector chooses T, and with $pr < s$, the noninspector chooses F. Clearly, some Loraxes must be NI types, since if all were I types, then all Thoraxes would cooperate, so a Lorax could gain from trusting rather than inspecting. If both I and NI are used, they must have equal payoffs, so $p^* = 1 - \delta/s$ if $pr > s$, and $p^* = \delta/(r-s)$ if $pr < s$. Suppose $s/r > \delta/(r-s)$. Then, $p^* = 1 - \delta/s > 1 - (r-s)/r = s/r$ is consistent with $pr > s$. If $s/r < \delta/(r-s)$, then we must have $p^* = \delta/(r-s) > s/r$, which implies $p^* = 1 - \delta/s$. This is only possible if $\delta/(r-s) = 1 - \delta/s$, or $s/r = \delta/(r-s)$, which is a contradiction. Hence, there is no mixed strategy

Nash equilibrium in this case. Assuming $pr > s$, the payoff to C for a second-mover is r, and the payoff to E is $1 - q$, so a mixed strategy equilibrium requires $q^* = 1 - r$.

d. Suppose $p > s/r$, so NI implies T for a Lorax. Then, (payoff to I)−(payoff to NI) = $(1-p)s-\delta$. For a Thorax, (payoff to C)−(payoff to E) = $r-(1-q)$. Then, the linear dynamic satisfies the equations

$$\dot{p} = b(q - (1 - r))$$
$$\dot{q} = -a\left[p - \left(1 - \frac{\delta}{s}\right)\right],$$

where $a, b > 0$ are rates of adjustment. The Jacobian at the fixed point $p^* = 1 - \delta/s$, $q^* = 1 - r$ is

$$J(p^*, q^*) = \begin{bmatrix} 0 & b \\ -a & 0 \end{bmatrix}.$$

This gives $\alpha = \text{Trace}(J(p^*, q^*))/2 = 0$, $\beta = \det(J(p^*, q^*)) = ab > 0$, $\gamma = \alpha^2 - \beta = -ab < 0$. Thus, the fixed point is a center.

11.20 Modeling Insiders and Outsiders

11.21 The Fundamental Theorem of Sociology

12

Markov Economies and Stochastic Dynamical Systems

12.1 Introduction

12.2 The Emergence of Money in a Markov Economy

12.3 Good Vibrations

12.4 Adaptive Learning

12.4.1 The Steady State of a Markov Chain

12.5 Adaptive Learning When Not All Conventions Are Equal

12.6 Adaptive Learning with Errors

12.7 Stochastic Stability

13

Learning Who Your Friends Are: Bayes' Rule and Private Information

13.1 Private Information

13.2 The Role of Beliefs in Games with Private Information

13.3 Haggling at the Bazaar

c. Let $\mu = \mathrm{P}(b_h|\text{refuse})$. Then by Bayes' Rule, with $x = P(\text{refuse}|b_h)$,

$$\begin{aligned}\mu &= \frac{\mathrm{P}(\text{refuse}|b_h)\mathrm{P}(b_h)}{\mathrm{P}(\text{refuse}|b_h)\mathrm{P}(b_h) + \mathrm{P}(\text{refuse}|b_l)\mathrm{P}(b_l)} \\ &= \frac{x\pi}{x\pi + 1 - \pi} \\ &= \frac{\pi}{\pi + \frac{1-\pi}{x}} \\ &\leq \pi.\end{aligned}$$

f. All buyers accept $p_1 = b_l$, so $p_1 < b_l$ is dominated. Any player who accepts p_1 for $b_l < p_1 < p^*$ accepts $p_1 = p^*$. No buyer accepts $p_1 > p^*$, since both high and low types prefer to wait until the second round and get $p_2 = b_l$. At $p_1 = p^*$, the payoffs to "accept" and "reject" are equal for a high-value buyer, since then $b_h - p_1 = \delta_b(b_h - b_l)$, so such a buyer accepts on round one.

i. Suppose the seller chooses $p_2 = b_l$. Since the seller's posterior probability for $\{b = b_l\}$ cannot be less than π (for the same reason as in the last problem) and since he would charge b_h in the one-shot game, he must charge $p_2 = b_h$. So suppose the seller chooses $p_2 = b_h$. Then the only undominated strategies on the first round are $p_1 = \delta_b b_h$ and $p_1 = b_l$. But if a buyer rejects $p_1 = \delta_b b_h$, he must be a low-value buyer, so it is not subgame perfect to charge $p_2 = b_h$.

13.4 Adverse Selection

13.5 A Market for Lemons

13.6 Choosing an Exorcist

13.7 A First-Price Sealed-Bid Auction

13.8 A Common Value Auction: The Winner's Curse

If $\tilde{v}$ is the measured value for any player, then

$$\mathrm{P}[\tilde{v} \le x|v] = \frac{x - v + a}{2a}.$$

Suppose your measured value, and hence bid, is x. The probability $F(x)$ that yours is the highest bid is just the probability that all $n-1$ other measured values are less than x, which is given by

$$F(x) = \left(\frac{x - v + a}{2a}\right)^{n-1}.$$

Your expected profit π is given by

$$\begin{aligned}
\pi &= \frac{1}{2a}\int_{v-a}^{v+a} (v - x)F(x)dx \\
&= \frac{1}{(2a)^n}\int_{v-a}^{v+a} (v - x)(x - v + a)^{n-1}dx \\
&= -a\frac{n-1}{n(n+1)}.
\end{aligned}$$

13.9 A Common Value Auction: Quantum Spin Decoders

a. If GE bids b, it wins if $b > (\alpha+\beta)y/2\beta$, or $2b\beta/(\alpha+\beta) > y$. GE's expected profit is thus

$$\int_0^{\frac{2b\beta}{\alpha+\beta}} (x + y - b)\frac{1}{\beta}dy = \frac{2b}{(\alpha+\beta)^2}\left[(\alpha+\beta)x - b\alpha\right].$$

If we equate the derivative of this expression with respect to b to zero and solve for b, we get $b = x(\alpha + \beta)/2\alpha$. A similar reasoning holds with respect to Martin Marietta. GE's expected profit is then $x^2/2\alpha$, which can be found by substitution. GE then wins if $x/\alpha > y/\beta$, which clearly has probability 1/2.

b. Each firm bids 1/2 of its known value, plus 1/2 of a random guess of the value it does not know. Note that when α/β is very small, kramel inhibitors are worth little. In this case GE knows the value of $\tilde{x}$ but the knowledge is practically worthless, so GE's bid is roughly 1/2 of a random sample from the uniform distribution on $[0, \beta]$. GE's expected profit is thus close to zero, although its probability of winning remains $1/2$. In this situation, Martin Marietta bids one half of the actual value of the beta-phase detector and can expect big profits.

c. If GE bids b having seen x, it wins if $b/\gamma > y$, so its expected profit is

$$\int_0^{\frac{b}{y}} (x + y - b)\frac{1}{\beta}dy.$$

Evaluating this integral and setting its derivative to zero gives $b = \gamma x/(2\gamma - 1)$. A similar argument holds for Martin Marietta.

d. Suppose the value is $\tilde{x} + \tilde{y} + \tilde{z}$ where $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are uniformly distributed on $[0, \alpha]$, $[0, \beta]$, and $[0, \gamma]$, respectively. Suppose the three bidders bid νx, ηy, and ζz, respectively. Bidder 1 then must determine ν given η and ζ. The expected profit of bidder 1 who sees x and bids b is given by

$$\int_0^{\frac{b}{\eta}} \int_0^{\frac{b}{\zeta}} (x + y + x - b)\frac{dy}{\eta}\frac{dz}{\zeta}.$$

Evaluating this integral, setting its derivative with respect to b equal to zero and solving for b, we get

$$b = F(\eta, \zeta)x,$$

where

$$F(\lambda, \mu) = \frac{4\lambda\mu}{3(2\lambda\mu - \lambda - \mu)},$$

so

$$\nu = F(\eta, \zeta).$$

If we substitute $\eta = \zeta = 5/3$, we see that $\nu^* = 5/3$, which proves the existence part of the assertion. For uniqueness, let (ν^*, η^*, ζ^*) be any solution. Note that if we repeat the above analysis to find η and ζ, we will find

$$\nu^* = F(\eta^*, \zeta^*)j, \eta^* = F(\zeta^*, \nu^*), \zeta^* = F(\eta^*, \nu^*).$$

If we substitute the first equation into the second and solve for η^*, we get

$$\eta^* = \frac{7\zeta^*}{6\zeta^* - 3}.$$

Now, substituting ν^* and $\eta*$ in the equation for ζ^*, we find $\zeta^* = 5/3$, so $\eta^* = 5/3$ and finally $\nu^* = 5/3$.

13.10 Predatory Pricing: Pooling and Separating Equilibria

If the equilibrium is separating, then the sane incumbent accommodates and earns $d_1(1+\delta)$. If the incumbent preyed in the first period, his earning would be $-p_1+\delta m$. Thus, a separating equilibrium requires (13.7). Clearly, this condition is also sufficient, when supplemented by the entrant's belief that the incumbent preys only if crazy, which is of course true if the equilibrium is separating.

If there is a pooling equilibrium, clearly (13.7) must be violated, or else the sane incumbent would prefer to accommodate. In addition, the sane firm must induce exit, since otherwise it would not be profitable to prey in the first period. Thus, we must have $\pi_2 d_2 - (1 - \pi_2)p_2 \leq 0$, which becomes

$$\pi_2 \leq \frac{p_2}{d_2 + p_2},$$

where π_2 is the entrant's posterior on the incumbent being sane if he preys in the first period. Clearly, $\pi_1 = \pi_2$ in this case, since all firms prey in the first period and hence no information is gained by observing the behavior of the incumbent. Thus, the condition for a pooling equilibrium is (13.8). Note that if the frequency of sane firms is sufficiently high (i.e., if π_1 is sufficiently large), the pooling equilibrium cannot occur even if the incumbent gains from driving an entrant out of the market.

Let us show that if equations (13.7) and (13.8) are both violated and if the entrant would enter the market if there were only one period, then we have a hybrid perfect Bayesian equilibrium, in which both parties randomize.

Let α be the probability that a sane incumbent preys in the first period, and let β be the probability that the entrant stays in the second period if the incumbent preys in the first period. The payoff to "stay" in the second period if the incumbent preys in the first is

$$-(1 - \pi_2)p_2 + \pi_2 d_2,$$

and this must equal the payoff to "drop," which is zero. Thus, we must have

$$\pi_2 = \frac{p_2}{d_2 + p_2}.$$

This is prima facie possible, since (13.8) is violated and $\pi_2 \leq \pi_1$.

To find β, note that the payoff to "prey" is

$$\beta(-p_1 + \delta d_1) + (1 - \beta)(-p_1 + \delta m),$$

and the payoff to "accommodate" is $d_1(1 + \delta)$. These must be equal, which gives (13.10). Note that β is always positive, and $\beta < 1$ precisely when the separating equilibrium condition (13.7) is violated.

But how do we get α? Obviously we must use Bayes' Rule. We have

$$\text{P[sane|prey]} = \frac{\text{P[prey|sane]P[sane]}}{\text{P[prey|sane]P[sane]} + \text{P[prey|crazy]P[crazy]}},$$

which becomes

$$\pi_2 = \frac{\alpha\pi_1}{\alpha\pi_1 + (1 - \pi_1)}.$$

Solving for α, we get (13.10). This is clearly positive and because (13.8) is violated, it is easy to show that $\alpha < 1$. Thus, we have a mixed strategy perfect Bayesian equilibrium.

13.11 Limit Pricing

a. Let $q = d(p)$ be demand for the good, $d' < 0$, and let c be constant marginal cost. Then profits are given by $\pi = d(p)(p - c)$ and the first-order order condition for π maximization is

$$\pi_p(p_m) = (p_m - c)d'(p_m) + d(p_m) = 0.$$

We take the total derivative of this with respect to c, treating $p_m = p_m(c)$ as an implicit function of c. We get

$$\pi_{pp}\frac{dp_m}{dc} + \pi_{pc} = 0.$$

Now $\pi_{pp} < 0$ by the second-order condition for profit maximization and $\pi_{pc} = (\partial/\partial c)((p_m - c)d'(p_m) + d(p_m)) = -d'(p_m) > 0$. Thus, $dp_m/d_c > 0$.

b. In a separating equilibrium, the low-cost firm does not want to pick the high-cost firm's price and viceversa. Moreover, the entrant only enters if the incumbent charges the high price. Let p^l be the price charged by the low-cost firm and suppose that if $p^l = p^l_m$, then the high-cost firm would find it profitable to charge this price too, so we would not have a separating equilibrium. Knowing this, when $c = c_l$, the incumbent might be willing to charge $p^l < p^l_m$ (limit pricing) to make it too costly for the high-cost firm to mimic this behavior.

Thus, for $c = c_h$ in a separating equilibrium, if δ is the discount factor we must then have $m^h + \delta d^h \geq m^h(p^l) + \delta m^h$, since charging p^h_m must be better than charging p^l. Also, for $c = c_l$, $m^l(p^l) + \delta m^l \geq m^l + \delta d^l$ must hold, since the low-cost firm must prefer charging p^l and being a monopolist in the second period (charging p^l_m) to charging p^h in the first period and being a duopolist in the second. Finally, limit pricing requires $p^l < p^l_m$, so at $c = c_h$, the incumbent would prefer to charge p^l_m if that convinced the entrant not to enter. Thus, we must have $m^h(p^l_m) + \delta m^h > m^h + \delta d^h$. In sum, for limit pricing to obtain (i.e., a separating equilibrium with $p^l < p^l_m$), we must have the following three inequalities holding simultaneously:

$$
\begin{aligned}
m^h - m^h(p^l) &\geq \delta(m^h - d^h) \\
\delta(m^h - d^h) &\geq m^h - m^h(p^l_m) \\
\delta(m^l - d^l) &\geq m^l - m^l(p^l).
\end{aligned}
$$

c. Here is the beginning of an example. It's not complete. Suppose demand is given by $q = d(p) = 1 - p$, so monopoly profits are $m = (1-p)(p-c) - f$, where c is marginal cost and f is fixed cost. The first-order condition for monopoly profits gives $p_m = (1+c)/2$, so $m(p_m) = (1-c)^2/4 - f$. Note that, as expected, monopoly price is an increasing function of marginal cost.

In the duopoly case, we assume a Cournot duopoly. Let q_1 and q_2 be the quantities of the two firms, so $q = q_1 + q_2$, and the price is $p = 1 - q = 1 - q_1 - q_2$. Duopoly profits for firm 1 are $d_1(q_1) = (1 - q_1 - q_2 - c)q_1 - f$ and the first-order condition for profit maximization is $q_1 = (1 - q_2 - c)/2$. Let c_2 be the cost of the entrant, so $d_2(q_2) = (1 - q_1 - q_2 - c_2)q_2 - f$ and the first-order condition gives $q_2 = (1 - q_1 - c_2)/2$. Solving simultaneously (the intersection of the offer curves), we get $q_1 = (1 + c_2 - 2c)/3$, $q_2 = (1 + c - 2c_2)/3$, so $p = 1 - q_1 - q_2 = (1 + c + c_2)/3$. Then $d_1 = (1 + c_2 - 2c)^2/9 - f$, $d_2 = (1 + c - 2c_2)^2/9 - f$. The condition $d^h_2 > 0 > d^l_2$ becomes $c^h - 2c_2 > 3\sqrt{f} - 1 > c^l - 2c_2$, which will hold for some f as long as $c^h > c^l$ and $c^h > 2c_2 - 1$.

Suppose $\delta = 1$ for simplicity. In a separating equilibrium, the l and h types do different things, and the entrant uses the signal to decide whether or not to enter. In the simplest case, an l incumbent would charge $p^l = p^l_m$ and the h incumbent would not want to follow suit. This requires that $m^h + d^h \geq$

$m^h(p^l_m)+m^h$, or $d^h \geq m^h(p^l_m)$, or $(1+c_2-2c^h)^2/9 \geq (1-c^l)(1+c^l-2c^h)/4$, or

$$c_2 \geq 3[(1-c^l)(1+c^l-2c^h)]^{1/2}/2 - 1 + 2c^h.$$

For concreteness, let's take $c^l = 0.1, c^h = 0.3$. Then this inequality becomes $c_2 \geq 0.6062$. Note that $c^h = 0.3 > 2c_2 - 1 = 0.2124$, so the condition $c^h > 2c_2 - 1$ holds as long as $c_2 < 0.65$. We must also show that the l firm prefers choosing p^l_m over p^h_m, which is obvious, since choosing p^l_m gives higher profits in the first period and keeps the competitor out. Conclusion: we have a separating equilibrium without limit pricing when $c^l = 0.1$, $c^h = 0.3$, $0.6062 \leq c_2 < 0.65$ and f is chosen so that $c^h - 2c_2 > 3\sqrt{f} - 1 > c^l - 2c_2$ is satisfied.

But suppose $c_2 < 0.6062$, which is likely for a potentially profitable firm, since these marginal costs are twice as high as c^h. The l firm can still signal the fact that it has low costs by charging $p^l < p^l_m$, if this is a strategy that the h firm would not follow. The h incumbent will not charge p^l if $m^h+d^h > m^h(p^l)+m^h$ (i.e., charging the monopoly price in the first period and facing a duopoly in the second is better than imitating the l firm in the first and being a monopoly in the second), or $d^h > m^h(p^l)$, which is $(1+c_2-2c^h)^2/9 > (1-p^l)(p^l-c^h)$. Treating this as an equality to determine the highest possible p^l with this property, we get

$$p^{l^2} - p^l(1+c^h) + (1 + c_2^2 + 4c^{h^2} + 2c_2 + 5c^h - 4c_2c^h)/9 = 0,$$

which is a quadratic equation. Using $c_2 = c^h = 0.3$, this becomes

$$p^{l^2} - 1.3p^l(1+c^h) + 0.354444 = 0,$$

whose relevant root is $p^l = 0.389125$. Note that $p^l_m = (1+c^l)/2 = 0.55$, so this is a significant case of limit pricing. Is this the best strategy for the l firm? By doing so, it receives $m(p^l) + m^l - 2f$, since the entrant does not enter in the second period. By charging p^l, the high-cost firm would do the same thing, so say the entrant enters (this depends on the entrant's subjective prior concerning the probability that the incumbent is an h or an l). Then the l firm earns $m^l + d^l - 2f$. Thus, limit pricing is Nash if $m(p^l) + m^l > m^l + d^l$, or $m(p^l) > d^l$. But $m(p^l) = (p^l - c^l)(1-p^l) = 0.176619$ and $d^l = 0.2025$, so this fails.

13.12 A Simple Limit-Pricing Model

If demand is $q = m - p$ and constant marginal cost is c, it is straightforward to check that the profit-maximizing price p^m, quantity q^m, and profit π^m for a monopolist are given by

$$q^m = \frac{m-c}{2}, \qquad p^m = \frac{m+c}{2}, \qquad \pi^m = \frac{(m-c)^2}{4}$$

provided $m \geq c$. Now suppose there are two firms in a Cournot duopoly, with costs c_1 and c_2, respectively. We assume c_1 and c_2 are common knowledge. It is straightforward to check that the optimal quantities q_i^d, prices p_i^d, and profits π_i^d for $i = 1, 2$ are given by

$$q_1^d = \frac{m + c_2 - 2c_1}{3}, \qquad q_2^d = \frac{m + c_1 - 2c_2}{3},$$

$$p^d = \frac{m + c_1 + c_2}{3}, \quad \pi_1^d = \frac{(m + c_2 - 2c_1)^2}{9}, \quad \pi_2^d = \frac{(m + c_1 - 2c_2)^2}{9}.$$

Also, the constraint that quantities be nonnegative requires $2c_1 \leq m + c_2$ and $2c_2 \leq m + c_1$. If one of the nonnegativity constraints is violated, it is easy to see that the higher-cost firm will not be in the market and the lower-cost firm will charge the monopoly price. For instance, suppose $c_1 > (m + c_2)/2$. If firm 2 produces the monopoly quantity, then $q_2 = (m - c_2)/2$, so

$$\begin{aligned}\frac{\partial \pi_1}{\partial q_1} &= m - 2q_1 - q_2 - c_1 = m - 2q_1 - (m - c_2)/2 - c_1 \\ &< m - 2q_1 - (m - c_2)/2 - (m + c_2)/2 = -2q_1,\end{aligned}$$

so the optimal response for firm 1 is to stay out of the market (i.e., to set $q_1 = 0$).

Suppose firm 1 is the incumbent (A) and firm 2 is the potential entrant (B), so by assumption $c_2 = c_h$. If firm 2 knew firm 1's costs, firm 2 would enter if and only if

$$\pi_2^d = (m + c_1 - 2c_h)^2/9 > f,$$

where f is the fixed entry cost. By assumption, this inequality must hold when $c_1 = c_h$ and fail when $c_1 = c_l$; i.e., we must have

$$(m + c_l - 2c_h)^2/9 \leq f \leq (m - c_h)^2/9.$$

In addition, it must not be profitable for A to pretend to be low-cost if A is in fact high-cost. We express this condition as follows. Let $(\pi_{mh}, \pi_{ml}, \pi_{dh}, \pi_{dl})$ be the monopoly profits of a high-cost firm, the monopoly profits of a low-cost firm, the duopoly profits of a high-cost firm facing a high-cost firm, and the duopoly profits of a low-cost firm facing a high-cost firm. Also, let π^* be the profits of a high-cost monopolist pretending to be a low-cost monopolist. For a separating equilibrium, a high-cost firm should not want to pretend to be low-cost. Assuming no discounting, this means we must have $\pi^* + \pi_{mh} < \pi_{mh} + \pi_{dh}$ and $(m + c_l - 2c_h)(m - c_l) < 4(m - c_h)^2/9$, or

$$\pi^* < \pi_{dh}.$$

This reduces to the condition

$$(m + c_l - 2c_h)(m - c_l) < \frac{4}{9}(m - c_h)^2.$$

Note that if c_l is close enough to c_h, this inequality fails.

14

When It Pays to Be Truthful: Signaling in Games with Friends, Adversaries, and Kin

14.1 Signaling as a Coevolutionary Process

14.2 A Generic Signaling Game

14.3 Introductory Offers

If a high-quality firm sells to a consumer in the first period at some price p_1, then in the second period the consumer will be willing to pay $p_2 = h$, since he knows the product is of high quality. Knowing that it can make a profit $h - c_h$ from a customer in the second period, a high-quality firm might want to make a consumer an "introductory offer" at a price p_1 in the first period that would not be mimicked by the low-quality firm, in order to reap the second-period profit.

If $p_1 > c_l$, the low-quality firm could mimic the high-quality firm, so the best the high-quality firm can do is to charge $p_1 = c_l$, which the low-quality firm will not mimic, since the low-quality firm cannot profit by doing so (it cannot profit in the first period, and the consumer will not buy the low-quality product in the second period). In this case, the high-quality firm's profits are $(c_l - c_h) + \delta(h - c_h)$. As long as these profits are positive, which reduces to $h > c_h + \delta(c_h - c_l)$, the high-quality firm will stay in business.

14.4 Web Sites (for Spiders)

a. In a truthful signaling equilibrium, strong spiders use saa and weak spiders use wra. To see this, note that strong spiders say they're strong and weak spiders say they're weak, by definition of a truthful signaling equilibrium. Weak spiders retreat against strong spiders because $d > 0$, and attack other weak spiders because $v - c_w > 0$. Strong spiders attack weak spiders if they do not withdraw, because $2v - b > 2v - c_s > v$.

b. If p is the fraction of strong spiders, then the expected payoff to a strong spider is $p(v - c_s) + 2(1 - p)v - e$, and the expected payoff to a weak spider is $(1 - p)(v - c_w)$. If these two are equal, then

$$p = \frac{v + c_w - e}{c_w + c_s}, \tag{A14.1}$$

which is strictly between 0 and 1 if and only if $e - c_w < v < e + c_s$.

c. In a truthful signaling equilibrium, each spider has expected payoff

$$\pi = \frac{(v - c_w)(c_s + e - v)}{c_w + c_s}. \tag{A14.2}$$

Suppose a weak spider signals that it is strong, and all other spiders play the truthful signaling equilibrium strategy. If the other spider is strong, it will attack and the weak spider will receive $-d$. If the other spider is weak it will withdraw, and the spider will gain $2v$. Thus, the payoff to the spider for a misleading communication is $-pd + 2(1 - p)v$, which cannot be greater than (A14.2) if truth-telling is Nash. Solving for d, we get

$$d \geq \frac{(c_s + e - v)(v + c_w)}{c_w - e + v}.$$

Can a strong spider benefit from signaling that it is weak? To see that it cannot, suppose first that it faces a strong spider. If it attacks the strong spider after signaling that it is weak, it gets the same payoff as if it signalled strong (since its opponent always attacks). If it withdraws against its opponent, it gets 0, which is less than the $v - c_s$ it gets by attacking. Thus, signaling weak against a strong opponent can't lead to a gain. Suppose the opponent is weak. Then signaling weak means that the opponent will attack. Responding by withdrawing, it gets 0; responding by attacking, it gets $2v - b$, since it always defeats its weak opponent. But if it had signalled strong, it would have earned $2v > 2v - b$. Thus, it never pays a strong spider to signal that it is weak.

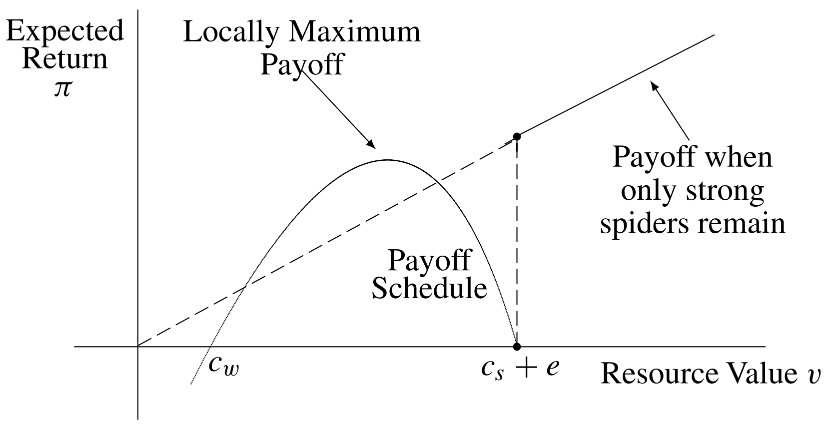
d. This result follows directly from equation (A14.2). This result occurs because higher e entails a lower fraction of strong spiders, from (A14.1). But weak spiders earn $(1 - p)(v - c_w)$, which is decreasing in p, and strong spiders earn the same as weak spiders in equilibrium.

e. We differentiate (A14.2) with respect to v, getting

$$\pi_v = \frac{c_w + c_s + e - 2v}{c_w + c_s}.$$

When $v = e - c_w > 0$, its smallest value, $\pi_v = 3c_w + c_s - e$, has an indeterminate sign. If it is negative, lowering v would increase payoff, as asserted. So suppose $\pi_v > 0$ at $v = e - c_w$. When $v = e + c_s$, which is its largest value, $\pi_v = c_w - c_s - e$, which is negative. Thus, there is an interior optimum for v in general.

Note that this optimum is only local; if we increase v beyond $e + c_s$ only strong spiders remain, and their payoff is now strictly increasing in v. Here is a picture, showing the discontinuity in the payoffs, once there are no weak spiders left. At this point the strong spiders no longer signal or attack, since all know all are strong. They each get v in this case.



14.5 Sex and Piety: The Darwin-Fisher Model of Sexual Selection

14.6 Biological Signals as Handicaps

14.7 The Shepherds Who Never Cry Wolf

The following payoffs are easy to derive:

$\pi_1(N,N) = p(1-a) + (1-p)(1-b); \quad \pi_2(N,N) = 1;$

$\pi_1(N,H) = p(1-a) + (1-p)(1-b); \quad \pi_2(N,H) = 1;$

$\pi_1(N,A) = 1; \quad \pi_2(N,A) = 1-d;$

$\pi_1(H,N) = p(1-a) + (1-p)(1-b) - pc; \quad \pi_2(H,N) = 1;$

$$\pi_1(H, H) = p(1-c) + (1-p)(1-b); \quad \pi_2(H, H) = p(1-d) + 1 - p;$$
$$\pi_1(H, A) = 1 - pc; \quad \pi_2(H, A) = 1 - d;$$
$$\pi_1(A, N) = p(1-a) + (1-p)(1-b) - c; \quad \pi_2(A, N) = 1;$$
$$\pi_1(A, H) = 1 - c; \quad \pi_2(A, H) = 1 - d;$$
$$\pi_1(A, A) = 1 - c; \quad \pi_2(A, A) = 1 - d.$$

Now the total payoff for shepherd 1 is $\pi_1^t = \pi_1 + k\pi_2$, and the total payoff for shepherd 2 is $\pi_2^t = \pi_1 + k\pi_1$. Substituting in numbers and forming the normal form matrix for the game, we get

	N	H	A
N	$\frac{19}{24}, \frac{37}{32}$	$\frac{19}{24}, \frac{37}{32}$	$\frac{21}{16}, \frac{7}{6}$
H	$\frac{95}{192}, \frac{793}{768}$	$\frac{47}{48}, \frac{829}{768}$	$\frac{65}{64}, \frac{267}{256}$
A	$\frac{19}{48}, \frac{571}{576}$	$\frac{11}{12}, \frac{577}{576}$	$\frac{11}{12}, \frac{577}{576}$

It is easy to see that (H, H) and (N, A) are Nash equilibria, and you can check that there is a mixed strategy equilibrium in which the threatened shepherd uses $\frac{1}{3}N + \frac{2}{3}H$ and the other shepherd uses $\frac{3}{5}H + \frac{2}{5}A$.

14.8 My Brother's Keeper

14.9 Honest Signaling among Partial Altruists

The payoff matrix for the encounter between a fisher observing a threatened fisher is as follows, where the first two lines are the payoffs to the individual players, and the third is the total payoff:

	Never Ask	Ask If Distressed	Always Ask
Never Help	$r(1-p)u$ $(1-p)u$ $(1+r)(1-p)u$	$r(1-p)u-rpt$ $(1-p)u-pt$ $(1+r)[(1-p)u-pt]$	$r(1-p)u-rt$ $(1-p)u-t$ $(1+r)[(1-p)u-t]$
Help If Asked	$r(1-p)u$ $(1-p)u$ $(1+r)(1-p)u$	$r[p(1-t)+(1-p)u]-pc$ $p(1-t)+(1-p)u$ $(1+r)[p(1-t)$ $+(1-p)u]-pc$	$r[p+(1-p)v-t]-c$ $p+(1-p)v-t$ $(1+r)[p+(1-p)v$ $-t]-c$
Always Help	$r[p+(1-p)v]-c$ $p+(1-p)v$ $(1+r)[p+(1-p)v]-c$	$r[p(1-t)+(1-p)v]-c$ $p(1-t)+(1-p)v$ $(1+r)[p(1-t)+$ $(1-p)v]-c$	$r[p+(1-p)v-t]-c$ $p+(1-p)v-t$ $(1+r)[p+(1-p)v$ $-t]-c$

The answers to the problem can be obtained in a straightforward manner from this matrix.

14.10 Educational Signaling

14.11 Education as a Screening Device

a. Given the probabilities (c), the wages (b) follow from

$$w_k = \mathrm{P}[a_h|e_k]a_h + \mathrm{P}[a_l|e_k]a_l, \qquad k = h, l. \tag{A14.3}$$

Then, it is a best response for workers to choose low education whatever their ability type, so (a) follows. Since both types choose e_l, the conditional probability $\mathrm{P}[a_l|e_l] = 1 - \alpha$ is consistent with the behavior of the agents, and since e_h is off the path of play, any conditional for $\mathrm{P}[a_l|e_h]$ is acceptable, so long as it induces a Nash equilibrium.

b. Assume the above conditions hold, and suppose c satisfies $a_l(a_h - a_l) < c < a_h(a_h - a_l)$. The wage conditions (b) follow from (14.3) and (c). Also, $a_h - c/a_h > a_l$, so a high-ability worker prefers to choose $e = 1$ and signal his true type, rather than choose e_l and signal his type as low ability. Similarly, $a_l > a_h - c/a_l$, so a low-ability worker prefers to choose e_l and signal his true type, rather than choose e_h and signal his type as high ability.

c. The wage conditions (b) follow from (14.3) and (c). Suppose $c < a_l(a_h - a_l)$. Then both high- and low-ability workers prefer to get education and the higher wage w_h rather than signal that they are low quality.

d. Let $\bar{e} = \alpha a_l(a_h - a_l)/c$, and choose $e^* \in [0, \bar{e}]$. Given the employer's wage offer, if a worker does not choose $e = e^*$ he might as well choose $e = 0$, since his wage in any case must be $w = a_l$. A low-ability worker then prefers to get education e^* rather than any other educational level, since $a_l \le \alpha a_h + (1 - \alpha)a_l - ce^*/a_l$. This is thus true for the high-ability worker, whose incentive compatibility constraint is not binding.

e. Consider the interval

$$\left[\frac{a_l(a_h - a_l)}{c}, \frac{a_h(a_h - a_l)}{c}\right].$$

If c is sufficiently large, this interval has a nonempty intersection with the unit interval [0, 1]. Suppose this intersection is $[e_{min}, e_{max}]$. Then, for $e^* \in [e_{min}, e_{max}]$, a high-ability worker prefers to acquire education e^* and receive the high wage $w = a_h$, while the low-ability worker prefers to receive $w = a_l$ with no education.

14.12 Capital as a Signaling Device

a. Given $p > 0$, choose k so that

$$1 > k(1 + \rho) > q + p(1 - q).$$

This is possible since $q + p(1 - q) < 1$. Then it is clear that the fraction q of good projects is socially productive. The interest rate r that a producer must offer then must satisfy

$$k(1 + \rho) = qk(1 + r) + (1 - q)kp(1 + r) = k(1 + r)[q + p(1 - q)],$$

so

$$r = \frac{1 + \rho}{q + p(1 - q)} - 1. \tag{A14.4}$$

The net profit of a producer with a good project is then

$$1 - k(1 + r) = \frac{q + p(1 - q) - k(1 + \rho)}{q + p(1 - q)} < 0,$$

so such producers will be unwilling to offer lenders an interest rate they are willing to accept. The same is clearly true of bad projects, so no projects get funded. Note that bad projects are not socially productive in this case, since $p - k(1+\rho) < p - (q + p(1-q)) = -q(1-p) < 0$.

b. Choose k so that $p < k(1+\rho) < q + p(1-q)$, which is clearly always possible. Then the fraction $1-q$ of bad projects are socially unproductive. The interest rate r must still satisfy equation (14.4), so the payoff to a successful project (good or bad) is

$$1 - k(1+r) = \frac{q + p(1-q) - k(1+\rho)}{q + p(1-q)} > 0,$$

so producers of both good and bad projects are willing to offer interest rate r, and lenders are willing to lend at this rate to all producers.

c. Let

$$k^p_{min} = \frac{p[1 - k(1+\rho)]}{(1-p)(1+\rho)}.$$

Note that since good projects are socially productive, $k^p_{min} > 0$. Suppose all producers have wealth $k^p > k^p_{min}$, and lenders believe that only a producer with a good project will invest k^p in his project. Then lenders will be willing to lend at interest rate ρ. If a producer invests k^p in his project and borrows $k - k^p$, his return is 1 and his costs are foregone earnings $k^p(1+\rho)$ and capital costs $(k - k^p)(1+\rho)$. Thus, his profit is

$$1 - k^p(1+\rho) - (k - k^p)(1+\rho) = 1 - k(1+\rho) > 0,$$

so such a producer is willing to undertake this transaction. If the producer with a bad project invests his capital k^p, his return is

$$p[1 - k(1+\rho)] - (1-p)k^p(1+\rho) < 0,$$

so he will not put up the equity. This proves the theorem.

15

Bosses and Workers, Landlords and Peasants, and Other Principal-Agent Models

15.1 Introduction to the Principal-Agent Model

15.2 Labor Discipline with Monitoring

15.3 Labor as Gift Exchange

a. Choosing w and N to maximize profits gives the first-order conditions

$$\pi_w(w, N) = [f'(eN)e' - 1]N = 0 \tag{A15.1}$$
$$\pi_N(w, N) = f'(eN)e - w = 0. \tag{A15.2}$$

Solving these equations gives the Solow condition.

b. The second partials are

$$\pi_{ww} = [f''Ne'^2 + f'e'']N < 0, \qquad \pi_{NN} = f''e^2 < 0,$$
$$\pi_{wN} = f''Nee' + f'e' - 1 = f''Nee' < 0.$$

It is easy to check that the second-order conditions are satisfied: $\pi_{ww} < 0$, $\pi_{NN} < 0$, and $\pi_{ww}\pi_{NN} - \pi_{wN}^2 > 0$.

c. To show that $dw/dz > 1$, differentiate the first-order conditions (A15.2) totally with respect to w and N:

$$\begin{aligned} \pi_{ww}\frac{dw}{dz} + \pi_{wN}\frac{dN}{dz} + \pi_{wz} = 0 \\ \pi_{Nw}\frac{dw}{dz} + \pi_{NN}\frac{dN}{dz} + \pi_{Nz} = 0. \end{aligned} \tag{A15.3}$$

Solving these two equations in the two unknowns dw/dz and dN/dz, we find

$$\frac{dw}{dz} = -\frac{\pi_{NN}\pi_{wz} - \pi_{Nw}\pi_{Nz}}{\pi_{NN}\pi_{wz} - \pi_{Nw}^2}. \tag{A15.4}$$

But we also calculate directly that

$$\pi_{wz} = -[f''Ne'^2 + f'e'']N = -\pi_{ww},$$
$$\pi_{Nz} = -f'e' - f''Nee' = -f'e' - \pi_{Nw}.$$

Substituting these values in (15.4), we get

$$\frac{dw}{dz} = 1 - \frac{\pi_{Nw} f'e'}{\pi_{NN}\pi_{wz} - \pi_{Nw}^2},$$

and the fraction in this expression is negative (the denominator is positive by the second-order conditions, while $\pi_{Nw} < 0$ and $f', e' > 0$).

Since $dw/dz > 1$, it follows from the chain rule that

$$\frac{de}{dz} = e'\left[\frac{dw}{dz} - 1\right] > 0,$$

$$\frac{dN}{dz} = \frac{-\pi_{wz} - \pi_{ww}\frac{dw}{dz}}{\pi_{wN}}$$
$$= \frac{\pi_{ww}}{\pi_{wN}}\left(1 - \frac{dw}{dz}\right) < 0 \qquad \text{[by (A15.3), (A15.4)]},$$

$$\frac{d\pi}{dz} = \frac{\partial\pi}{\partial w}\frac{dw}{dz} + \frac{\partial\pi}{\partial N}\frac{dN}{dz} + \frac{\partial\pi}{\partial z} = \frac{\partial\pi}{\partial z} = -f'e'N < 0.$$

15.4 Labor Discipline with Profit Signaling

b. Treat w_H as a function of w_L, and differentiate the participation constraint, getting

$$p_h u'(w_H)\frac{dw_H}{dw_L} + (1 - p_h)u'(w_L) = 0.$$

Thus,

$$\frac{dw_H}{dw_L} = -\frac{1 - p_h}{p_h}\frac{u'(w_L)}{u'(w_H)} < -\frac{1 - p_h}{p_h} < 0. \qquad \text{(A15.5)}$$

The second inequality (which we use later) holds because $w_L < w_H$, so if the agent is strictly risk averse, u' is decreasing. The participation constraint is thus decreasing. Now take the derivative of equation (A15.5), getting

$$\frac{d^2 w_H}{dw_L^2} = -\frac{1 - p_h}{p_h}\left[\frac{u''(w_H)}{u'(w_H)} - \frac{u'(w_L)u''(w_H)}{u'(w_H)^2}\frac{dw_H}{dw_L}\right] > 0.$$

Thus, the participation constraint is convex.

c. We differentiate the incentive compatibility constraint $u(w_H) = u(w_L) +$ constant, getting

$$u'(w_H)\frac{dw_H}{dw_L} = u'(w_L),$$

so $dw_H/dw_L > 1 > 0$ and the incentive compatibility constraint is increasing. Differentiate again, getting

$$u''(w_H)\frac{dw_H}{dw_L} + u'(w_H)\frac{d^2w_H}{dw_L^2} = u''(w_L).$$

Thus

$$u'(w_H)\frac{d^2w_H}{dw_L^2} = u''(w_L) - u''(w_H)\frac{dw_H}{dw_L} < u''(w_L) - u''(w_H) < 0,$$

and the constraint is concave.

d. The slope of the iso-cost line is $|dw_H/dw_L| = (1 - p_h)/p_h$, which is less than the slope of the participation constraint, which is

$$|(1 - p_h)u'(w_L)/p_h u'(w_H)|,$$

by equation (A15.5).

15.5 Peasants and Landlords

a. First, w_H and w_L must satisfy a *participation constraint:* w_H and w_L must be sufficiently large that the peasant is willing to work at all. Suppose the peasant's next-best alternative gives utility z. Then the landowner must choose w_H and w_L so that the peasant's expected utility is at least z:

$$p_h u(w_H) + (1 - p_h)u(w_L) - d_h \geq z. \quad \text{(PC)}$$

Second, w_H and w_L must satisfy an *incentive compatibility constraint:* the payoff (i.e., the expected return) to the peasant for working hard must be at least as great as the payoff to not working hard. Thus, we must have

$$p_h u(w_H) + (1 - p_h)u(w_L) - d_h \geq p_l u(w_H) + (1 - p_l)u(w_L) - d_l.$$

We can rewrite this second condition as

$$[u(w_H) - u(w_L)](p_h - p_l) \geq d_h - d_l. \quad \text{(ICC)}$$

b. The problem is to minimize $p_h w_H + (1 - p_h) w_L$ subject to PC and ICC. This is the same as maximizing $-p_h w_H - (1 - p_h) w_L$ subject to the same constraints, so we form the Lagrangian

$$\begin{aligned}\mathcal{L}(w_H, w_L, \lambda, \mu) = -p_h w_H - (1 - p_h) w_L \\ + \lambda[p_h u(w_H) + (1 - p_h) u(w_L) - d_h - z] \\ + \mu[(u(w_H) - u(w_L))(p_h - p_l) - (d_h - d_l)].\end{aligned}$$

The first-order conditions can be written:

$$\mathcal{L}_H = 0, \mathcal{L}_L = 0, \qquad \lambda, \mu \geq 0;$$

$$\text{if } \lambda > 0 \text{ then the PC holds with equality;}$$

$$\text{if } \mu > 0, \text{ then the ICC holds with equality.}$$

But we have

$$\begin{aligned}\mathcal{L}_H &= -p_h + \lambda p_h u'(w_H) + \mu u'(w_H)(p_h - p_l) = 0, \\ \mathcal{L}_L &= -1 + p_h + \lambda(1 - p_h) u'(w_L) - \mu u'(w_L)(p_h - p_l) = 0.\end{aligned}$$

We show that $\lambda = 0$ is impossible. Assume the contrary, so $\lambda = 0$. Then, by adding the two first-order conditions, we get

$$\mu(u'(w_H) - u'(w_L))(p_h - p_l) = 1,$$

which implies $u'(w_H) > u'(w_L)$, so $w_H < w_L$ (by declining marginal utility of income). This is, of course, silly, since the peasant will not want to work hard if the wage given high profits is less than the wage given low profits! More formally, ICC implies $u(w_H) > u(w_L)$, so $w_H > w_L$. It follows that our assumption that $\lambda = 0$ was contradictory and hence $\lambda > 0$. So PC holds as an equality.

We now show that $\mu = 0$ is impossible. Assume the contrary, so $\mu = 0$. Then the first-order conditions $\mathcal{L}_H = 0$ and $\mathcal{L}_L = 0$ imply $u'(w_H) = 1/\lambda$ and $u'(w_L) = 1/\lambda$. Since $u'(w_H) = u'(w_L) = 1/\lambda$, $w_H = w_L$ (because u' is strictly decreasing). This also is impossible by the ICC. Hence $\mu > 0$ and the ICC holds as an equality.

c. Now suppose the landlord has concave utility function v, with $v' > 0$ and $v'' < 0$. The peasant is risk neutral, so we can assume her utility function is $u(w, d) = w - d$, where w is income and d is effort. The assumption that high effort produces a surplus means that the following social optimality (SO) condition holds:

$$p_h H + (1 - p_h)L - d_h > p_l H + (1 - p_l)L - d_l,$$

or

$$(p_h - p_l)(H - L) > d_h - d_l. \tag{SO}$$

The landlord wants to maximize

$$p_h v(H - w_H) + (1 - p_h)v(L - w_L)$$

subject to the participation constraint

$$p_h w_H + (1 - p_h)w_L - d_h \geq z \tag{\textbf{PC}}$$

and the incentive compatibility constraint, which as before reduces to

$$(p_h - p_l)(w_h - w_L) \geq d_h - d_l. \tag{\textbf{ICC}}$$

We form the Lagrangian

$$\begin{aligned}\mathcal{L} = p_h v(H - w_H) &+ (1 - p_h)v(L - w_L) \\ &+ \lambda(p_h w_H + (1 - p_h)w_L - d_h - z) \\ &+ \mu((p_h - p_l)(w_h - w_L) - (d_h - d_l)),\end{aligned}$$

so the first-order conditions are $\partial\mathcal{L}/\partial w_H = \partial\mathcal{L}/\partial w_L = 0$, $\lambda, \mu \geq 0$, $\lambda > 0$ or the PC holds as an equality, and $\mu > 0$ or the ICC holds as an equality. The conditions $\partial\mathcal{L}/\partial w_H = \partial\mathcal{L}/\partial w_L = 0$ can be written as

$$\frac{\partial\mathcal{L}}{\partial w_H} = -p_h v'(H - w_H) + \lambda p_h + \mu(p_h - p_l) = 0 \tag{FOC1}$$

$$\mu(p_h - p_l) = 0. \tag{FOC2}$$

We first show that the PC holds as an equality by showing that $\lambda = 0$ is impossible. Suppose $\lambda = 0$. Then (FOC) gives

$$-p_h v'(H - w_H) + \mu(p_h - p_l) = 0$$
$$-(1 - p_h)v'(L - w_L) - \mu(p_h - p_l) = 0.$$

If $\mu > 0$, this says that $v'(L - w_L) < 0$, which is impossible. If $\mu = 0$, this says that $v'(L-w_L) = 0$, which is also impossible. Thus, $\lambda = 0$ is impossible, and the PC holds as an equality.

Now we can rewrite FOC1 and FOC2 as

$$v'(H - w_H) - \lambda = \mu(p_h - p_l)/p_h \quad \text{(A15.6)}$$
$$v'(L - w_L) - \lambda = -\mu(p_h - p_l)/(1 - p_h). \quad \text{(A15.7)}$$

If $\mu > 0$, then $v'(H - w_H) > v'(L - w_L)$, so $H - w_H < L - w_L$, or $H - L < w_H - w_L$. But $\mu > 0$ implies that the ICC holds as an equality, so $(p_h - p_l)(w_H - w_L) = d_h - d_l$, and $d_h - d_l < (p_h - p_l)(H - L)$ from SO, implying $H - L > w_H - w_L$. This is a contradiction and hence $\mu = 0$; i.e., there is no optimum in which the ICC holds as an equality!

What is an optimum? Well, if $\mu = 0$, equations (A15.6) and (A15.7) imply that $H - w_h = L - w_L$, since v' is strictly increasing (because the landlord is risk averse). This means the landlord gets a fixed rent, which proves the theorem.

15.6 Mr. Smith's Car Insurance

a. If he is careful, the value is $(1/2)\ln(1201) - \epsilon$, and if he is careless the value is $(1/4)\ln(1201)$. Being careful is worthwhile as long as $(1/2)\ln(1201) - \epsilon \geq (1/4)\ln(1201)$, or $\epsilon \leq \ln(1201)/4 = 1.77$.
b. If he buys the insurance, his utility is $\ln(1201 - x)$ if he is careless, and $\ln(1201 - x) - \epsilon$ if he is careful. Thus, he will not be careful. The probability of theft is then 75%, and since the insurance company must give a fair lottery to Mr. Smith, $x = (0.75)1200 = 900$.
c. The expected value of the car plus insurance is $\ln(1201 - 900) = \ln(301) = 5.707$.
d. This is preferable to being careless without insurance, which has value

$$(1/4)\ln(1201) = 1.77.$$

The value of being careful without insurance has value $(1/2)\ln(1201) - \epsilon < (1/2)\ln(1201) = 3.55$, so Mr. Smith should buy the insurance policy whether or not he would be careful without it.
e. Since Mr. Smith is careful without insurance, we know $\epsilon \leq 1.77$. Since the insurance company's lottery is fair, we have $x = (1200 - z)/2 = 600 - z/2$

if Mr. Smith is careful, and $x = 3(1200 - z)/4 = 900 - 3z/4$ if Mr. Smith is careless. If Mr. Smith is careless, the value of car plus insurance is

$$\frac{1}{4}\ln(1201 - x) + \frac{3}{4}\ln(1201 - z - x) = \frac{1}{4}\ln\left(301 + \frac{3z}{4}\right) + \frac{3}{4}\ln\left(301 - \frac{z}{4}\right).$$

The derivative of this with respect to z is

$$(3/16)/(301 + 3z/4) - (3/16)/(301 - z/4) < 0,$$

so the optimal deductible is zero. Then $x = 900$, and the value of car plus insurance is $\ln(1201 - 900) = \ln(301)$. If Mr. Smith is careful, then the value of car plus insurance is

$$\frac{1}{2}\ln(1201 - x) + \frac{1}{2}\ln(1201 - z - x) = \frac{1}{2}\ln\left(601 + \frac{z}{2}\right) + \frac{1}{2}\ln\left(601 - \frac{z}{2}\right).$$

The derivative of this with respect to z is $(1/8)/(601 + z/2) - (1/8)/(601 - z/2) < 0$, so again the optimal deductible is zero. But z must be sufficiently large that Mr. Smith wants to be careful, or the insurance company will not be willing to issue the insurance at the low rate $x = 600 - z/2$. To make taking care worthwhile, we must have

$$\frac{1}{2}\ln\left(601 - \frac{z}{2}\right) + \frac{1}{2}\ln\left(601 - \frac{z}{2}\right) - \epsilon \geq \frac{1}{4}\ln\left(601 + \frac{z}{2}\right) + \frac{3}{4}\ln\left(601 - \frac{z}{2}\right),$$

$\ln(601 + z/2) - \ln(601 - z/2) \geq 4\epsilon$. The minimum z satisfying this is when the equality holds, and we have

$$z = \frac{4(e^{4\epsilon} - 1)}{e^{4\epsilon} + 1} > 0.$$

f. With insurance, the expected value of the lottery for Mr. Smith is unchanged, but the risk has decreased. Since he is risk averse, he is better off with the insurance than without.

15.7 A Generic One-Shot Principal-Agent Game

a. Intuitively, we argue as follows. Suppose the principal wants to induce the agent to choose action a_k. If the participation constraint is not binding, we can reduce all the payments $\{w_{kj}\}$ by a small amount without violating either the participation or the incentive compatibility constraints. Moreover, if all of the incentive compatibility constraints are nonbinding, then the payoff system $\{w_{kj}\}$ is excessively risky, in the sense that the various $\{w_{kj}\}$ can be "compressed" around their expected value without violating the incentive compatibility constraint.

Formally, we form the Lagrangian,

$$\mathcal{L} = \pi(a_k) - \mathbf{E}_k w_k + \lambda[\mathbf{E}_k u(w_k) - d(a_k) - u_o] + \sum_{\substack{i=1 \\ i \neq k}}^{n} \mu_i \{[\mathbf{E}_k u(w_k) - d(a_k)] - [\mathbf{E}_i u(w_k) - d(a_i)]\},$$

where $\mathbf{E}_i$ means take the expectation with respect to probabilities $\{p_{i1}, \ldots, p_{im}\}$, λ is the Lagrangian multiplier for the participation constraint, and μ_i is the Lagrangian multiplier for the ith incentive compatibility constraint. Writing the expectations out in full (in "real life" you wouldn't do this, but it's valuable for pedagogical purposes), we get

$$\mathcal{L} = \pi(a_k) - \sum_{j=1}^{m} p_{kj} w_{kj} + \lambda \left[\sum_{j=1}^{m} p_{kj} u(w_{kj}, a_k) - u_o \right] + \sum_{i=1}^{n} \mu_i \left[\sum_{j=1}^{m} p_{kj} u(w_{kj}, a_k) - \sum_{j=1}^{m} p_{ij} u(w_{kj}, a_i) \right].$$

The Kuhn-Tucker conditions for the problem assert that at a maximum,

1. $\frac{\partial \mathcal{L}}{\partial w_{kj}} = 0$ for $j = 1, \ldots, m$.
2. $\lambda, \mu_1, \ldots, \mu_n \geq 0$.
3. If $\lambda > 0$, then the participation constraint is binding (i.e., holds as an equality).
4. If $\mu_i > 0$ for some $i = 1, \ldots, n$, then the incentive compatibility constraint holds for action a_i.

In our case, we have

$$\frac{\partial \mathcal{L}}{w_{kj}} = -p_{kj} + \lambda p_{kj} u'(w_{kj}, a_k)$$
$$+ \sum_{i=1}^{n} \mu_i (p_{kj} - p_{ij}) u'(w_{kj}, a_k) = 0 \quad j = 1, \ldots, m.$$

Collecting terms, we have, for $j = 1, \ldots, m$,

$$\lambda p_{kj} + \sum_{i=1}^{n} \mu_i (p_{kj} - p_{ij}) = \frac{p_{kj}}{u'(w_{kj}, a_k)}. \tag{A15.8}$$

Now we sum this equation from $j = 1$ to m, noting that $\sum_j p_{ij} = 1$ for all i, getting

$$\lambda = \sum_{j=1}^{m} \frac{p_{kj}}{u'(w_{kj}, a_k)} > 0,$$

which proves that the participation constraint is binding.

1. To see that at least one incentive compatibility constraint is binding, suppose all the μ_i are zero. Then equation (A15.8) gives $\lambda = 1/u'(w_{kj}, a_k)$ for all j. If the agent is risk averse, this implies all the w_{kj} are equal (since risk aversion implies strictly concave preference, which implies strictly monotonic marginal utility), which means a_k must be the agent's most preferred action.
2. If the parties could write an enforceable contract, then only the participation constraint would be relevant. To induce the agent to perform a_i, the principal would pay w_i^*, where $u(w_i^*, a_i) = u_o$. The principal will then choose action a_l such that $\pi(a_l) - w_l^*$ is a maximum.

   Suppose the optimal action when no contract can be written is a_k, and the wage structure is $w_k(s)$. The participation constraint is binding, so $\mathbf{E}_k u(w_k) = d(a_k) + u_o$. Since a_k is not the agent's unconstrained preferred action (by assumption), $w_k(s)$ is not constant, and since the agent is strictly risk averse, we have

   $$u(w_k^*, a_k) = u_o = \mathbf{E}u(w_k, a_k) < u(\mathbf{E}_k w_k, a_k),$$

   so $w_k^* < \mathbf{E}_k w_k$. Since $\pi(a_l) - w_l^* \geq \pi(a_k) - w_k^* > \pi(a_k) - \mathbf{E}_k w_k$, we are done.

b. Suppose again that a_l is the principal's optimal choice when an enforceable contract can be written. Then a_l maximizes $\pi(a_i) - w_i^*$ where $u(w_i^*, a_i) = u_o$. If the agent is risk neutral, suppose the agent receives $\pi(a)$ and pays the principal the fixed amount $\pi(a_l) - w_l^*$. We can assume $u(w, a) = w + d(a)$ (why?). The agent then maximizes

$$\begin{aligned}\pi(a_i) - d(a_i) &= u(\pi(a_i)) - u(w_i^*) - u_o \\ &= \pi(a_i) - w_i^* - u_o,\end{aligned}$$

which of course occurs when $i = l$. This proves the theorem. ■

16

Bargaining

16.1 Introduction

16.2 The Nash Bargaining Model

Let $x = u(\pi_1^{\mathrm{o}})$ and $y = v(\pi_2^{\mathrm{o}})$. We maximize $(u(\pi_1) - x)(v(\pi_2) - y)$ subject to $g(\pi_1, \pi_2) = 0$. The solution turns out to depend directly on the second-order conditions for constrained maximization. We form the Lagrangian

$$\mathcal{L}(\pi_1, \pi_2, \lambda) = (u(\pi_1) - x)(v(\pi_2) - y) - \lambda g(\pi_1, \pi_2).$$

The first-order conditions for this constrained optimization problem are then

$$\begin{aligned}\mathcal{L}_{\pi_1} &= u'(\pi_1)(v(\pi_2) - y) - \lambda g_{\pi_1} = 0\\ \mathcal{L}_{\pi_2} &= v'(\pi_2)(u(\pi_1) - x) - \lambda g_{\pi_2} = 0.\end{aligned}$$

Now let's differentiate the two first-order conditions and the constraint $g(\pi_1, \pi_2) = 0$ totally with respect to player 1's fallback x:

$$\begin{aligned}\mathcal{L}_{\pi_1\pi_1}\frac{d\pi_1}{dx} + \mathcal{L}_{\pi_1\pi_2}\frac{d\pi_2}{dx} + \mathcal{L}_{\pi_1\lambda}\frac{d\lambda}{dx} + \mathcal{L}_{\pi_1 x} &= 0\\ \mathcal{L}_{\pi_2\pi_1}\frac{d\pi_1}{dx} + \mathcal{L}_{\pi_2\pi_2}\frac{d\pi_2}{dx} + \mathcal{L}_{\pi_2\lambda}\frac{d\lambda}{dx} + \mathcal{L}_{\pi_2 x} &= 0\\ g_{\pi_1}\frac{d\pi_1}{dx} + g_{\pi_2}\frac{d\pi_2}{dx} &= 0.\end{aligned}$$

Now $\mathcal{L}_{\pi_1 x} = 0$, $\mathcal{L}_{\pi_2 x} = -v'(\pi_2)$, so we have

$$\begin{bmatrix} \mathcal{L}_{\pi_1\pi_1} & \mathcal{L}_{\pi_1\pi_2} & -g_{\pi_1} \\ \mathcal{L}_{\pi_2\pi_1} & \mathcal{L}_{\pi_2\pi_2} & -g_{\pi_2} \\ -g_{\pi_1} & -g_{\pi_2} & 0 \end{bmatrix} \begin{bmatrix} \frac{d\pi_1}{dx} \\ \frac{d\pi_2}{dx} \\ \frac{d\lambda}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ v'(\pi_2) \\ 0 \end{bmatrix}$$

Note that the matrix is the bordered Hessian of the problem, which has positive determinant D by the second-order conditions. If we solve for $d\pi_1/dx$ by using Cramer's rule, we get $d\pi_1/dx = g_{\pi_1} g_{\pi_2} v'/D$. The first-order conditions imply that g_{π_1} and g_{π_2} have the same sign, so $d\pi_1/dx > 0$.

16.3 Risk Aversion and the Nash Bargaining Solution

Suppose first that $v(x) = h(u(x))$, where h is concave. Then

$$v'(x) = h'(u(x))u'(x)$$

and

$$v''(x) = h''(u(x))u'^2(x) + h'(u(x))u''(x),$$

so

$$\lambda_v(x) = \lambda_u(x) - \frac{h''(u(x))u'(x)}{h'(u(x))} > \lambda_u(x).$$

Thus, $v(x)$ is uniformly more risk averse than $u(x)$.

Now assume $v(x)$ is uniformly more risk averse than $u(x)$. Since u is monotone increasing, u^{-1} exists, so we may define $h(y) = v(u^{-1}(y))$. Clearly, $h(0) = 0$. Also, $h(y)$ is increasing, since the identity $u(u^{-1}(y)) = y$ implies

$$\frac{d}{dy}u^{-1}(y) = \frac{1}{u'(u^{-1}(y))},$$

and hence,

$$\frac{dh}{dy} = \frac{d}{dy}v(u^{-1}(y)) = \frac{v'(u^{-1}(y))}{u'(u^{-1}(y))} > 0.$$

Differentiating again, we get

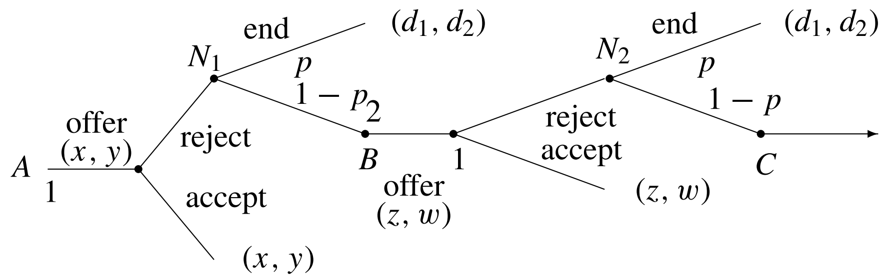
$$\frac{d^2h}{dy^2} = \frac{\lambda_u(u^{-1}(y)) - \lambda_v(u^{-1}(y))}{v'(u^{-1}(y))u'^2(u^{-1}(y))} < 0,$$

where the last inequality follows from the fact that $v(x)$ is more risk averse than $u(x)$.

16.4 Bargaining with Two-Sided Outside Options

16.5 Rubinstein Bargaining and Nash Bargaining

The game tree is as follows.



Let x be the maximum payoff to agent 1 at C. Then the maximum payoff to agent 1 at N_2 is $pd_1 + (1-p)x$, so agent 2 must offer agent 1 at least $\delta(pd_1 + (1-p)x)$ at B, so the minimum share of agent 2 at B is $g(\delta(pd_1 + (1-p)x))$. Thus, the minimum share of agent 2 at N_1 is $(1-p)g(\delta(pd_1 + (1-p)x)) + pd_2$, so agent 1 must offer agent 2 at least

$$\delta(1-p)g(\delta(pd_1 + (1-p)x)) + \delta pd_2$$

at A$'$. Thus, the maximum agent 1 can get at A is

$$g^{-1}(\delta(1-p)g(\delta(pd_1 + (1-p)x)) + \delta pd_2).$$

This must equal x, so we have

$$\delta(1-p)g(\delta(pd_1 + (1-p)x)) + \delta pd_2 = g(x).$$

The solution to this equation is the Rubinstein bargaining equilibrium. If we let $\delta = 1$, this becomes

$$(1-p)g(pd_1 + (1-p)x) + pd_2 = g(x),$$

which we can rewrite as

$$(1-p)\frac{g(x + p(d_1 - x)) - g(x)}{p(d_1 - x)} = \frac{g(x) - d_2}{d_1 - x}.$$

If we take the limit as $p \to 0$, this becomes

$$g(x) = d_2 - g'(x)(x - d_1).$$

But this is just the solution to the Nash bargaining problem

$$\max_x (x - d_1)(g(x) - d_2),$$

as can be seen by differentiating this function and setting it to zero.

16.6 Zeuthen Lotteries and the Nash Bargaining Solution

Suppose the theorem is false, and there are $s, s^* \in S$ such that s^* is a Zeuthen equilibrium, but

$$(u_1(s) - u_1(d))(u_2(s) - u_2(d)) > (u_1(s^*) - u_1(d))(u_2(s^*) - u_2(d)).$$

Then either $u_1(s) > u_1(s^*)$ or $u_2(s) > u_2(s^*)$; assume the former. Let $p^* = (u_1(s^*) - u_1(d))/(u_1(s) - u_1(d))$, so $0 \leq p^* < 1$, and define $p_\epsilon = p^* + \epsilon$ for $0 < \epsilon < 1 - p^*$. Then $p_\epsilon(u_1(s) - u_1(d)) > u_1(s^*) - u_1(d)$ so $p_\epsilon u_1(s) + (1 - p_\epsilon)u_1(d) > u_1(s^*)$, which means that the expected utility from (p_ϵ, s) is greater than the (expected) utility from s; i.e., $(p_\epsilon, s) \succ_1 s$. By assumption, this means that $(p_\epsilon, s^*) \succeq_2 s$. In expected utility terms, this becomes $p_\epsilon u_2(s^*) + (1 - p_\epsilon)u_2(d) \geq u_2(s)$, or $p_\epsilon(u_2(s^*) - u_2(d)) \geq u_2(s) - u_2(d)$. Since this is true for arbitrarily small $\epsilon > 0$, it is true for $\epsilon = 0$, giving $p^*(u_2(s^*) - u_2(d)) \geq u_2(s) - u_2(d)$. But by the definition of p^*, we then have $((u_1(s^*) - u_1(d))/(u_1(s) - u_1(d)))(u_2(s^*) - u_2(d)) \geq u_2(s) - u_2(d)$, or $(u_1(s^*) - u_1(d))(u_2(s^*) - u_2(d)) \geq (u_1(s) - u_1(d))(u_2(s) - u_2(d))$, which contradicts our assumption.

16.7 Bargaining with Fixed Costs

a. Clearly, Mutt cannot do better by deviating. Suppose Jeff deviates on some round and then returns to his prescribed behavior. If any round other than the first is involved, Jeff cannot improve his payoff, because the game is already terminated! To deviate on the first round, Jeff must reject. This costs him c_2. He returns to his prescribed strategy by offering Mutt $1 - c_1$, which Mutt accepts. This leaves Jeff with $c_1 - c_2 < 0$, so Jeff is worse off.
b. Let l_2 be the least Jeff can get in any subgame perfect Nash equilibrium when he goes first. Since Mutt can get m_1 when it is his turn, Jeff must offer Mutt at most $m_1 - c_1$, which leaves Jeff with at least $l_1 = 1 - m_1 + c_1 \leq 1$. Suppose $l_1 \geq c_2$. The most that Mutt can make is then $1 - (l_1 - c_2)$, so

$$m_1 = 1 - (l_1 - c_2) = m_1 - c_1 + c_2,$$

which implies $c_1 = c_2$, which is false. If $l_1 < c_2$, then Mutt offers Jeff 0, and $m_1 = 1$, which is also a contradiction.

c. In this case, $l_2 = 1$, since Jeff can offer Mutt zero when it is Jeff's turn. But then Mutt can offer Jeff $1 - c_2$, so Mutt earns at least $c_2 > c_1$, which is a contradiction.

16.8 Bargaining with Incomplete Information

a. The seller chooses p_2 to maximize

$$pP[v > p] = pP\left[\frac{v}{v_2} > \frac{p}{v_2}\right] = p\left(1 - \frac{p}{v_2}\right),$$

from which the result follows by setting the derivative to zero and solving for p_2.

b. If $v \geq p_2$, the buyer will accept the second-round offer, earning $v - p_2$, which is worth $\delta(v - p_2)$ in the first period. The buyer thus rejects the first offer if and only if $v - p_1 < \delta(v - p_2)$, which gives us the desired condition. If $v < p_2$, p_1 is accepted if and only if $v \geq p_1$.

c. Suppose $v \geq p_1$ or $v \geq p_2$. If p_1 is rejected, then $v < (p_1 - \delta p_2)/(1 - \delta)$, and since $p_2 = v_2/2$, we can solve for v_2.

d. Suppose $v \geq p_1$ or $v \geq p_2$. The payoff to the seller from offering p in the first period is

$$pP\left[v - p \geq \delta\left(v - \frac{p}{2-\delta}\right)\right].$$

This evaluates directly to $p(1 - 2p/v_1(2 - \delta))$, which achieves its maximum when $p = v_1(2 - \delta)/4$. The rest follows by substitution.

17

Probability and Decision Theory

17.1 Basic Set Theory and Mathematical Notation

17.2 Probability Spaces

17.3 DeMorgan's Laws

17.4 Interocitors

17.5 The Direct Evaluation of Probabilities

17.6 Probability as Frequency

The only hard part is answering questions like "what is the probability of rolling a 4 before rolling 2, 7, or 12?" Let's call this probability $q(4)$. Since there are three ways to roll a 4 (13, 22, 31) out of 36 possible rolls, the probability of rolling a 4 is $p(4) = 3/36$. The ways of rolling a 2, 7, or 12 are (11, 61, 52, 43, 34, 25, 16, 66), so the probability of "crapping out" is 8/36. This gives us the equation

since if you don't crap out, the probability of rolling a 4 before crapping out is still $q(4)$. We can solve this for $q(4)$, getting $q(4) = 9/31$.

17.7 Sampling

We do only part b. *A* die has six possible numbers. Throwing six dice is like sampling one die six times with replacement. Thus, there are $6^6 = 46656$ ordered configurations of the 6 dice. There are 6 outcomes in which all the faces are the same. Thus, the probability is $6/46656 = 0.0001286$.

17.8 Self-presentation

17.9 Social Isolation

There are n^r equiprobable arrangements of passengers in cars; i.e., the number of ordered samples with replacement. By assigning different passengers to different cars, we are looking at the subset of ordered samples without replacement (after one passenger enters a car, the car cannot be entered by another passenger). Thus, there are $n!/(n-r)!$ arrangements with no two passengers in the same car. The probability of the event A = {no two passengers in the same car} is thus $\mathrm{P}[A] = n!/(n-r)!n^r$.

17.10 Aces Up

There are 52 ways to choose the first card, and 51 ways to choose the second card. Since we don't care about the order of the choices, there are $52 \times 51/2 = 26 \times 51$ different ways to choose two cards from the deck. There are 4 ways to choose the first ace, and 3 ways to choose the second, and since we don't care about the order, there are $4 \times 3/2 = 6$ ways to choose a pair of aces. Thus, the probability of choosing a pair of aces is $6/(26 \times 51) = 1/(13 \times 17) = 1/221 = 0.0045248 = 0.45248\%$.

17.11 Mechanical Defection

This is sampling 2 times without replacement from a set of 7 objects. There are $7!/(7-2)! = 7 \times 6 = 42$ such samples. How many of these are two non-defective machines? How many samples of two are there from a population of 5 (the number of non-defectives)? The answer is $5!/(5-2)! = 5 \times 4 = 20$. Thus, the probability is $20/42 = 0.4762$.

17.12 Double Orders

There are n^r equiprobable assignments of customers to firms; i.e., the number of ordered samples with replacement. By assigning different customers to different firms, we are looking at the subset of ordered samples without replacement (after a customer places an order with a firm, the firm cannot receive another order). Thus, there are $n!/(n-r)!$ arrangements with no two customers using the same firm. The probability of the event $A = \{$no two customers use the same firm$\}$ is thus $\mathrm{P}[A] = n!/(n-r)!n^r$.

17.13 Combinations and Sampling

17.14 Mass Defection

The number of ways of selecting 10 items from a batch of 100 items equals the number of combinations of 100 things taken 10 at a time, which is 100!/10!90!. If the batch is accepted, all of the 10 items must have been chosen from the 90 non-defective items. The number of such combinations of ten items is 90!/10!80!. Thus, the probability of accepting the batch is

$$\begin{aligned}\frac{(90!/10!80!)}{(100!/10!90!)} &= 90!90!/80!100! \\ &= \frac{90 \times 89 \times \ldots \times 81}{100 \times 99 \times \ldots 91},\end{aligned}$$

which is approximately 36.8%.

17.15 An Unlucky Streak

There are $\binom{n}{b}$ ways of choosing b items from a lot of n items. There are $\binom{k}{a}$ ways of choosing a items from the k defective items, and $\binom{n-k}{b-a}$ ways of picking the other $b-a$ good items from the $n-k$ good items. Thus, there are $\binom{k}{a}\binom{n-k}{b-a}$ ways of choosing a sample with exactly a defective items. The probability of choosing exactly a defective items is thus

$$\frac{\binom{k}{a}\binom{n-k}{b-a}}{\binom{n}{b}}.$$

Note that the probability of getting *at least c* defective items is the sum of the above expression for $a = c, a = c+1, \ldots$

17.16 House Rules

Here is an equivalent game: you *ante* \$1000 and choose a number. The house rolls the three dice, and pays you \$2000 for one match, \$3000 for two matches, and \$4000 for three matches. The probability of one match is

$$\binom{3}{1}\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{75}{216},$$

the probability of two matches is

$$\binom{3}{2}\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{15}{216},$$

and the probability of three matches is 1/216. The expected payoff is thus

$$2\frac{75}{216}+3\frac{15}{216}+4\frac{1}{216}=\frac{199}{216}=0.9213$$

times \$1000, or \$921.30. Thus, you can expect to lose \$78.70 every time you play.

17.17 The Powerball Lottery

The number of ways to pick the five white balls and the one red Powerball is

$$\binom{49}{5}=\frac{49\times 48\times 47\times 46\times 45}{5\times 4\times 3\times 2}\times 42=80089128.$$

Thus, the probability of winning is $p = 1/80089128$. The probability of no winner in $n = 138500000$ tries is then

$$\left(1-\frac{1}{80089128}\right)^{138500000}\approx 0.1774.$$

If your calculator chokes on this calculation, try the formula

$$(1-p)^n = e^{n\ln(1-p)}.$$

Thus, the probability of at least one winner is 82.26%.

The probability that the first bettor wins and nobody else wins is $p(1-p)^{n-1}$, and the probability that exactly one bettor wins is n times this, or

$$n\times p\times(1-p)^{n-1}\approx 30.68\%.$$

17.18 The Addition Rule for Probabilities

17.19 Die, Die!

Let A be the event "the two dice are the same" and let B be the event "the two dice add up to eight." There are six ways the dice can be the same, so $\mathrm{P}[A] =$

$6/36 = 1/6$. There are 5 ways the dice can add up to eight (2 and 6, 3 and 5, 4 and 4, 5 and 3, 6 and 2), so $\mathrm{P}[B] = 5/36$. There is one way the dice can be the same and add up to eight (4 and 4), so $\mathrm{P}[A \cap B] = 1/36$. Thus, $\mathrm{P}[A \cup B] = 1/6 + 5/36 - 1/36 = 15/54 = 5/18$.

17.20 Les Cinq Tiroirs

We depict the whole event space as a rectangle with six pieces. Piece A, which consists of 20% of the space, represents the event "the object is not in any drawer."

A	D1	16%
	D2	16%
	D3	16%
	D4	16%
20%	D5	16%

The other five events, D1, D2, D3, D4, and D5, represent the event where the object is in one of the drawers. Since these are equally likely, each such event represents (1-0.2)/5 = 16% of the space.

The probability of D1, which we may write $\mathrm{P}[D1]$ is, of course 16%. The probability of D2 given not D1 is $\mathrm{P}[D2|D1^c]$. We can evaluate this by

$$\mathrm{P}[D2|D1^c] = \frac{\mathrm{P}[D2 \wedge D1^c]}{\mathrm{P}[D1^c]} = \frac{\mathrm{P}[D2]}{1 - 0.16} = 0.16/0.84 \approx 19\%.$$

The probability of D3 given not D1 or D2 is $\mathrm{P}[D3|D1^c \wedge D2^c]$. We can evaluate this by

$$\mathrm{P}[D3|D1^c \wedge D2^c] = \frac{\mathrm{P}[D3 \wedge D1^c \wedge D2^c]}{\mathrm{P}[D1^c \wedge D2^c]} = \frac{\mathrm{P}[D3]}{1 - 0.16 - 0.16}$$
$$= 0.16/0.68 \approx 23.5\%.$$

You can check that the probability of finding the object in the fourth drawer, given that it was not in any previous drawer, is $0.16/0.52 = 30.77\%$, and the probability that it is in the fifth drawer given that it is neither of the first four is $0.16/0.36 = 44.44\%$. So the probability of finding the object rises from drawer 1 to drawer 5.

What about the probability of not finding the object? Let N be the event "the object is in none of the drawers." the $\mathrm{P}[N] = 0.2$. What is $\mathrm{P}[N|D1^c]$, the probability it is none of the drawers if it is not in the first drawer. Well, by definition of conditional probability,

$$\mathrm{P}[N|D1^c] = \frac{\mathrm{P}[N \wedge D1^c]}{\mathrm{P}[D1^c]} = \frac{\mathrm{P}[N]}{\mathrm{P}[D1^c]} = 0.2/0.84 = 23.81\%.$$

The probability the object is in none of the drawers if it is found not to be in either of the first two is, similarly (do the reasoning!) $0.2/0.68 = 29.41\%$. It is easy now to do the rest of the problem (do it!).

17.21 Equity in Tennis

For men, there are two ways the match can end in three sets–either player 1 or player 2 wins the first three. This outcome thus occurs with probability $2 \times 1/8 = 1/4$. If the match ends in four sets, the looser could have won the first, the second, or the third, and the loser could be player 1 or player 2. Hence there are six ways the match can end in four sets, which thus occurs with probability $6 \times 1/16 = 3/8$. The probability the match lasts five sets is thus $1 - 1/4 - 3/8 = 3/8$. The expected number of sets played is thus $3 \times 1/4 + 4 \times 3/8 + 5 \times 3/8 = 4.125$. For women, there are two ways the match can end in two sets, which thus occurs with probability $2 \times 1/4 = 1/2$. Thus the probability of a three set match is also 1/2. The expected number of sets is thus $2 \times 1/2 + 3 \times 1/2 = 2.5$.

17.22 A Guessing Game

17.23 Bayes' Rule

17.24 Drug Testing

We have $\mathrm{P}[A] = 1/20$ and $\mathrm{P}[\mathrm{Pos}|A] = \mathrm{P}[\mathrm{Neg}|A^c] = 19/20$. Thus,

$$\begin{aligned}\mathrm{P}[A|\mathrm{Pos}] &= \frac{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A]}{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A] + \mathrm{P}[\mathrm{Pos}|A^c]\,\mathrm{P}[A^c]} \\ &= \frac{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A]}{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A] + (1 - \mathrm{P}[\mathrm{Neg}|A^c])\mathrm{P}[A^c]} \\ &= \frac{(19/20)(1/20)}{(19/20)(1/20) + (19/20)(1/20)} = 1/2.\end{aligned}$$

We can answer the problem without using Bayes' Rule just by counting. Suppose we test 10000 people (the number doesn't matter). Then $10000 \times 0.05 = 500$ use drugs (on average), of whom $500 \times 0.95 = 475$ test positive (on average). But 9500 do not use drugs (again, on average), and $9500 \times (1 - 0.95) = 475$ also test positive (on average). Thus of the 950 ($= 475 + 475$) who test positive, exactly 50% use drugs (on average).

17.25 A Bolt Factory

Let D be the event "bolt is defective." Then the three probabilities are $\mathrm{P}[A|D] = 125/273$, $\mathrm{P}[B|D] = 140/273$,and $\mathrm{P}[C|D] = 8/273$.

17.26 Color Blindness

The probability the person is male is $100/105$.

17.27 Urns

17.28 The Monty Hall Game

17.29 The Logic of Murder and Abuse

17.30 The Principle of Insufficient Reason

17.31 The Greens and the Blacks

Let A be the event "A bridge hand contains at least two aces." Let B be the event "A bridge hand contains at least one ace." Let C be the event "A bridge hand contains the ace of spades."

Then $\mathrm{P}[A|B]$ is the probability that a hand contains two aces if it contains one ace and hence is the first probability sought. Also $\mathrm{P}[A|C]$ is the probability a hand contains two aces if it contains the ace of spades, which is the second probability sought. By Bayes' Rule,

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[AB]}{\mathrm{P}[B]} = \frac{\mathrm{P}[A]}{\mathrm{P}[B]},$$

and

$$\mathrm{P}[A|C] = \frac{\mathrm{P}[AC]}{\mathrm{P}[C]}.$$

Clearly, $\mathrm{P}[C] = 0.25$, since all four hands are equally likely to get the ace of spades.

To calculate $\mathrm{P}[B]$, note that the total number of hands with no aces is the number of ways to take 13 objects from 48 (the 52 cards minus the four aces), which is $\binom{48}{13}$.

The probability of a hand having at least one ace is then

$$\mathrm{P}[B] = \frac{\binom{52}{13} - \binom{48}{13}}{\binom{52}{13}} = 1 - \frac{39 \times 38 \times 37 \times 36}{52 \times 51 \times 50 \times 49} = 0.6962.$$

The probability of at least two aces is the probability of at least one ace minus the probability of exactly one ace. We know the former, so let's calculate the latter.

The number of hands with exactly one ace is four times $\binom{48}{12}$, since you can choose the ace in one of four ways, and then choose any combination of 12 cards from the 48 non-aces. But

$$\frac{4 \times \binom{48}{12}}{\binom{52}{13}} = \frac{39 \times 38 \times 37}{51 \times 50 \times 49} \approx 0.4388,$$

which is the probability of having exactly one ace. The probability of at least two aces is thus

$$\mathrm{P}[A] = .6962 - .4388 = .2574$$

(to four decimal places).

Now $\mathrm{P}[AC]$ is the probability of two aces including the ace of spades. The number of ways to get the ace of spades plus one other ace is calculated as follows: take the ace of spades out of the deck, and form hands of twelve cards. The number of ways of getting no aces from the remaining cards is $\binom{48}{12}$, so the number of hands with one other ace is $\binom{51}{12} - \binom{48}{12}$. The probability of two aces including the ace of spades is thus

$$\frac{\binom{51}{12} - \binom{48}{12}}{\binom{52}{13}} = .1402.$$

Thus, $\mathrm{P}[AC] = .1402$. We now have

$$\mathrm{P}[A|C] = \frac{\mathrm{P}[AC]}{\mathrm{P}[C]} = \frac{.1402}{.25} = .5608 > \frac{\mathrm{P}[AB]}{\mathrm{P}[B]} = \frac{.2574}{.6962} = .3697.$$

17.32 Laplace's Law of Succession

Suppose there are n balls in the urn, and assume the number of white balls is uniformly distributed between 0 and n. Let A_k be the event "there are k white balls," and let B_{rm} be the event "of m balls chosen with replacement, r are white." Then $\mathrm{P}[A_k] = 1/(n+1)$, and by Bayes' Rule we have

$$\mathrm{P}[A_k|B_{rm}] = \frac{\mathrm{P}[B_{rm}|A_k]\mathrm{P}[A_k]}{\mathrm{P}[B_{rm}]}.$$

Now it is easy to check that

$$\mathrm{P}[B_{rm}|A_k] = \binom{m}{r}\left(\frac{k}{n}\right)^r\left(1-\frac{k}{n}\right)^{m-r}$$

and

$$\mathrm{P}[B_{rm}] = \sum_{k=0}^{n}\mathrm{P}[A_k]\mathrm{P}[B_{rm}|A_k]. \qquad \text{(A17.1)}$$

The probability of choosing a white ball on the next draw is then

$$\sum_{k=0}^{n}\left(\frac{k}{n}\right)\mathrm{P}[A_k|B_{rm}] = \sum_{k=0}^{n}\frac{k\mathrm{P}[B_{rm}|A_k]}{n(n+1)\mathrm{P}[B_{rm}]}$$
$$= \frac{1}{(n+1)\mathrm{P}[B_{rm}]}\binom{m}{r}\sum_{k=0}^{n}\left(\frac{k}{n}\right)^{r+1}\left(1-\frac{k}{n}\right)^{m-r}.$$

To approximate this expression, note that if n is large, equation (17.1) is a Riemann sum representing the integral

$$\mathrm{P}[B_{rm}] \approx \frac{1}{n+1}\binom{m}{r}\int_0^1 x^r(1-x)^{m-r}$$
$$= \frac{1}{(n+1)(m+1)},$$

where the integral is evaluated by integration by parts r times. Replacing m by $m+1$ and r by $r+1$ in the above expression, we see that equation (A17.2) is approximately

$$\frac{1}{(n+1)\mathrm{P}[B_{rm}]}\frac{m!}{r!(m-r)!}\frac{(r+1)!(m-r)!}{(m+2)(m+1)!} = \frac{r+1}{m+2}.$$

17.33 The Brain and Kidney Problem

Let A be the event "the jar contains two brains" and let B be the event "the mad scientist pulls out a brain." Then $\mathrm{P}[A] = \mathrm{P}[A^c] = 1/2$, $\mathrm{P}[B|A] = 1$, and $\mathrm{P}[B|A^c] = 1/2$. Then from Bayes' Rule, the probability that the remaining blob is a brain is $\mathrm{P}[A|B]$, which is given by

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[B|A]\mathrm{P}[A]}{\mathrm{P}[B|A]\mathrm{P}[A] + \mathrm{P}[B|A^c]\mathrm{P}[A^c]} = \frac{1/2}{1/2 + (1/2)(1/2)} = 2/3.$$

17.34 Sexual Harassment on the Job

If you are harassed in the first 10 years, you will quit. The probability that this does *not* happen is $p_o = (0.95)^{10} \simeq 0.598737$, so with probability $1 - p_o$ you will be harassed exactly once. After 10 years you will not quit if harassed, and since you remain for 20 more years, the expected number of harassments is one. The conclusion is that you expect to be harassed on the job exactly once.

17.35 The Value of Eyewitness Testimony

Let G be the event "Cab that hit JD was green," let B be the event "cab that his JD was blue," let WB be the event "witness records seeing blue cab," and finally, let WG be the event "witness records seeing green cab." We have $\mathrm{P}[G] = 85/100 = 17/20$, $\mathrm{P}[B] = 15/100 = 3/20$, $\mathrm{P}[WG|G] = \mathrm{P}[WB|B] = 4/5$, $\mathrm{P}[WB|G] = \mathrm{P}[WG|B] = 1/5$. Then Bayes' Rule yields

$$\mathrm{P}[B|WB] = \frac{\mathrm{P}[WB|B]\mathrm{P}[B]}{\mathrm{P}[WB|B]\mathrm{P}[B] + \mathrm{P}[WB|G]\mathrm{P}[G]},$$

which evaluates to 12/29.

17.36 The End of the World

17.37 Bill and Harry

Let A = "Bill's statement is true" and let B = "Harry says Bill's statement is true." Also let A^c = "Bill's statement is false." Then $\mathrm{P}[A] = \mathrm{P}[B] = 1/3$, and

$P[A^c] = 2/3$. From Bayes' Rule,

$$\begin{aligned} P[A|B] &= \frac{P[B|A]P[A]}{P[B|A]P[A] + P[B|A^c]P[A^c]} \\ &= \frac{(1/3)(1/3)}{(1/3)(1/3) + (2/3)(2/3)} = \frac{1}{5}, \end{aligned}$$

so the probability that Bill's statement is true given Harry's affirmation is 20%.

17.38 When Weakness Is Strength

a. Clearly, if c aims at anybody, it should be a. If c hits a, then his payoff is $\pi_c(bc)$. But clearly $\pi_c(bc) = \pi_c(cb)/5$, since b misses c with probability $1/5$, and if he misses, it is c's turn to shoot. But

$$\pi_c(cb) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{5}\pi_c(cb),$$

since either c wins immediately (with probability 1/2), or if he misses (with probability 1/2) he gets another turn with probability 1/5. This gives $\pi_c(cb) = 5/9$, so $\pi_c(bc) = 1/9$. We conclude that if c hits a, he survives with probability 1/9. What if c shoots in the air? Clearly, a will shoot at b rather than c, and b will shoot at a rather than c. Thus, a and b will trade shots until one is gone, and then it is c's turn to shoot. Suppose it is a who remains. Then the probability that c survives is $\pi_c(ca)$, which is clearly 1/2 (c shoots at a and if he misses, he loses the game). If b remains, the probability c wins is $\pi_c(cb)$, which we have already calculated to be $5/9$. Since both 5/9 and 1/2 are greater than 1/9, c does much better by shooting in the air.

b. Suppose first that a goes before b, so the order is cab or acb. Since c fires in the air, a will take out b for sure, so $\pi_c(cab) = \pi_c(acb) = 1/2$, since then c has exactly one chance to shoot at a. Also, $\pi_a(cab) = \pi_a(acb) = 1/2$ and $\pi_b(cab) = \pi_b(acb) = 0$. Now suppose b goes before a. Then b gets one shot at a, and hits with probability 4/5. Thus,

$$\begin{aligned} \pi_c(cba) = \pi_c(bca) &= \frac{4}{5}\pi_c(cb) + \frac{1}{5}\pi_c(ca) \\ &= \frac{4}{5} \times \frac{5}{9} + \frac{1}{5} \times \frac{1}{2} \\ &= \frac{49}{90}. \end{aligned}$$

Moreover, $\pi_a(cba) = \pi_a(bca) = (1/5)(1/2) = 1/10$, and $\pi_b(cba) = \pi_b(bca) = (4/5)(4/9) = 16/45$. We then have $\pi_a = (1/2)(1/2 + 1/10) = 3/10$, $\pi_b = (1/2)(16/45) = 8/45$, and $\pi_c = (1/2)(1/2 + 49/90) = 23.5/45$. Notice that $\pi_c > \pi_a > \pi_b$.

c. If the marksman who shoots next is randomly chosen, it is still true that a and b will shoot at each other until only one of them remains. However, clearly c now prefers to have a one-on-one against b rather than against a, so c will shoot at a if given the chance. Now

$$\pi_a(ab) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{5}\pi_a(ab),$$

so $\pi_a(ab) = 5/9$ and $\pi_b(ab) = 4/9$. Similar reasoning gives $\pi_a(ac) = 2/3$ and $\pi_c(ac) = 1/3$. Finally,

$$\pi_b(bc) = \frac{1}{2}\left(\frac{4}{5} + \frac{1}{5}\pi_b(bc)\right) + \frac{1}{2} \times \frac{1}{2}\pi_b(bc),$$

from which we conclude $\pi_b(bc) = 8/13$ and $\pi_c(bc) = 5/13$. Now clearly $\pi_a[a] = \pi_a(ac) = 2/3$, $\pi_b[a] = 0$, and $\pi_c[a] = 1/3$. Similarly, it is easy to check that

$$\begin{aligned}
\pi_b[b] &= (4/5)\pi_b(bc) + (1/5)\pi_b \\
\pi_a[b] &= (1/5)\pi_a \\
\pi_c[b] &= (4/5)pi_c(bc) + (1/5)\pi_c \\
\pi_c[c] &= (1/2)\pi_c(bc) + (1/2)\pi_c \\
\pi_b[c] &= (1/2)\pi_b(bc) + (1/2)\pi_b \\
\pi_a[c] &= (1/2)\pi_a.
\end{aligned}$$

Moving to the final calculations, we have

$$\pi_b = \frac{1}{3}\left[0 + \frac{4}{5}\pi_b(bc) + \frac{1}{5}\pi_b + \frac{1}{2}\pi_b(bc) + \frac{1}{2}\pi_b\right].$$

We can solve this for π_b, getting $\pi_b = 24/69$. The similar equation for marksman a is

$$\pi_a = \frac{1}{3}\left[\frac{2}{3} + \frac{1}{5}\pi_a + \frac{1}{2}\pi_a\right],$$

which gives $\pi_a = 20/69$. Finally,

$$\pi_c = \frac{1}{3}\left[\frac{1}{3} + \frac{4}{5}\pi_c(bc) + \frac{1}{5}\pi_c + \frac{1}{2}\pi_c(bc) + \frac{1}{2}\pi_c\right],$$

which gives $\pi_c = 25/69$. Clearly, $\pi_c > \pi_b > \pi_a$, so the meek inherit the earth.

17.39 The Uniform Distribution

17.40 Markov Chains

17.40.1 Andrei Andreyevich's Two-Urn Problem